QUANTUM MECHANICS*

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Preface

I consider in this book a formulation of Quantum Mechanics, which is often abbreviated as QM. Usually QM is formulated based on the notion of time and space, both of which are thought *a priori* given quantities or notions. However, when we try to define the notion of velocity or momentum, we encounter a difficulty as we will see in chapter 1. The problem is that if the notion of time is given *a priori*, the velocity is definitely determined when given a position, which contradicts the uncertainty principle of Heisenberg.

We then set the basis of QM on the notion of position and momentum operators as in chapter 2. Time of a local system then is defined approximately as a ratio $|x|/|v|$ between the space coordinate x and the velocity v, where $|x|$, etc. denotes the absolute value or length of a vector x. In this formulation of QM, we can keep the uncertainty principle, and time is a quantity that does not have precise values unlike the usually supposed notion of time has.

The feature of local time is that it is a time proper to each local system, which is defined as a finite set of quantum mechanical particles. We now have an infinite number of local times that are unique and proper to each local system.

Based on the notion of local time, the motion inside a local system is described by the usual Schrödinger equation. We investigate such motion in a given local system in part II. This is a usual quantum mechanics.

After some excursion of the investigation of local motion, we consider in part III the relative relation or motion between plural local systems. We regard each local system's center of mass as a classical particle. Then as the relative coordinate inside a local system is independent of its center of mass, we can set an arbitrary rule on the relation among those centers of mass of local systems. We adopt the principles of general relativity as the rules that govern the relations of plural local systems. By the reason that the center of mass and the inner coordinate are independent, we can combine quantum mechanics and general relativity consistently.

We give an approximate Hamiltonian that explains partially the usual relativistic quantum mechanical phenomena in chapter 9.

In the final part IV, we consider some contradictory aspect of mathematics in chapter 10. Although this does not give directly that mathematics is inconsistent, this will give an introduction to the next chapter 11, where starting with the contradictory nature of the semantics of set theory in the sense that if we consider all sentences of set theory, they are contradictory, we regard that the Universe that is described by ourselves is of contradictory nature, and can be described as a superposition of all possible, infinite number of waves. As this is the state of the Universe, the Universe is described as a stationary state describing a superposition of all waves. We then give a formulation of the Universe and local systems inside it, in the form of a theory described by Axiom 1 to Axiom 5 in chapter 11. In the final chapter 12, we will prove that there is at least one Universe wave function ϕ in which all local systems have local motions and thus local times. This concludes our formulation of Quantum Mechanics.

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Notation

We here explain some notations which will be used in the text. $C^k(R^m)$ $(k = 0, 1, 2, \cdots, \infty)$ is a space of k times continuously differentiable functions $f(x)$ of $x \in R^m$. $C_0^k(R^m)$ is a subspace of $C^k(R^m)$ whose element $f \in C^k(R^m)$ has compact support in R^m. In particular, $C_0^\infty(R^m)$ is a space of infinitely differentiable functions on R^m with compact support. We also use the notation $C_0^\infty(G)$ for a region G in R^m to denote the space of functions with continuous derivatives up to order k with the support contained in G. $\mathcal{S} = \mathcal{S}(R^m)$ denotes a space of rapidly decreasing functions f on R^m. Namely $f \in \mathcal{S}$ means that f is an infinitely differentiable function satisfying

$$\sup_{x \in R^m} \left| |x|^k \partial_x^\alpha f(x) \right| < \infty \tag{1}$$

for all integers $k = 0, 1, 2, \cdots$ and multi-indices $\alpha = (\alpha_1, \cdots, \alpha_m)$, where each α_j is a non-negative integer and

$$\partial_x = (\partial/\partial x_1, \cdots \partial/\partial x_m), \quad \partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_m)^{\alpha_m}. \tag{2}$$

The most important notion is the Hilbert space $L^2(R^m)$ with $m = 1, 2, \cdots$. It is a space of functions $f(x)$ on R^m satisfying

$$\|f\| = (f, f)^{1/2} < \infty, \tag{3}$$

where the inner product is given by

$$(f, g) = \int_{R^m} f(x)\overline{g(x)}dx. \tag{4}$$

Concretely it is obtained by a completion of $\mathcal{S}$ or of $C_0^\infty(R^m)$ with respect to the norm (3). Along with this Hilbert space we use weighted L^2 space: $L_s^2 = L_s^2(R^m)$ $(s \in R^1)$, which is a completion of $\mathcal{S}$ with respect to the norm

$$\|f\|_s = \|f\|_{L_s^2} = \left(\int_{R^m} |f(x)|^2 \langle x \rangle^{2s} dx \right)^{1/2}, \tag{5}$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$. $L_s^2(R^m)$ is also a Hilbert space with the inner product:

$$(f, g)_s = (f, g)_{L_s^2} = \int f(x)\overline{g(x)}\langle x \rangle^{2s} dx. \tag{6}$$

$\mathcal{F}$ denotes Fourier transformation from $\mathcal{S}$ onto itself:

$$\mathcal{F}f(\xi) = (2\pi)^{-m/2} \int e^{-i\xi x} f(x) dx. \tag{7}$$

$\mathcal{F}$ is extended to a unitary operator from $L^2(R^m)$ onto itself:

$$\|\mathcal{F}f\| = \|f\|. \tag{8}$$

We define Sobolev space $H^s = H^s(R^m)$ of order $s \in R^1$ as the Fourier image of $L^2_s(R^m)$. Thus it is a completion of $\mathcal{S}$ with respect to the norm

$$\|f\|_{H^s} = \left(\int_{R^m} |\langle D_x \rangle^s f(x)|^2 dx \right)^{1/2}. \tag{9}$$

Here

$$\langle D_x \rangle^s = \mathcal{F}^{-1} \langle \xi \rangle^s \mathcal{F}, \tag{10}$$

where $\langle \xi \rangle^s$ denotes a multiplication operator by $\langle \xi \rangle^s$ in $L^2(R^m_\xi)$. Namely its domain $\mathcal{D}(\langle \xi \rangle^s)$ is a set of functions f satisfying

$$\langle \xi \rangle^s f(\xi) \in L^2(R^m_\xi) \tag{11}$$

and its value applied to $f \in \mathcal{D}(\langle \xi \rangle^s)$ is $\langle \xi \rangle^s f(\xi) \in L^2(R^m)$. $H^s(R^m)$ is also a Hilbert space with the inner product:

$$(f, g)_{H^s} = \int_{R^m} \langle D_x \rangle^s f(x) \overline{\langle D_x \rangle^s g(x)} dx. \tag{12}$$

We further use weighted Sobolev space $H^s_\delta(R^m)$ ($\delta \in R^1$). This is a completion of $\mathcal{S}$ with respect to the norm

$$\|f\|_{H^s_\delta} = \left(\int_{R^m} |\langle D_x \rangle^s f(x)|^2 \langle x \rangle^{2\delta} dx \right)^{1/2}, \tag{13}$$

and is a Hilbert space with the inner product

$$(f, g)_{H^s_\delta} = \int_{R^m} \langle D_x \rangle^s f(x) \overline{\langle D_x \rangle^s g(x)} \langle x \rangle^{2\delta} dx. \tag{14}$$

S^{m-1} denotes the unit sphere of R^m with surface element $d\omega$. $L^2(S^{m-1})$ is a space of the functions $\varphi(\omega)$ satisfying

$$\|\varphi\|_{L^2(S^{m-1})} = \left(\int_{S^{m-1}} |\varphi(\omega)|^2 d\omega \right)^{1/2} < \infty. \tag{15}$$

$L^2(S^{m-1})$ becomes a Hilbert space with the inner product

$$(\varphi, \psi)_{L^2(S^{m-1})} = \int_{S^{m-1}} \varphi(\omega) \overline{\psi(\omega)} d\omega. \tag{16}$$

We remark that $L^2_s(R^m)$ and $L^2_{-s}(R^m)$ are dual spaces each other with respect to the inner product of $L^2(R^m)$. Similarly $H^s(R^m)$ and $H^s_\delta(R^m)$ are dual spaces of $H^{-s}(R^m)$ and $H^{-s}_{-\delta}(R^m)$, respectively. $L^2(R^m)$ is a dual space of itself. We denote by $B(\mathcal{H}_1, \mathcal{H}_2)$ the

Banach space of bounded operators from a Hilbert space $\mathcal{H}_1$ into another Hilbert space $\mathcal{H}_2$. The notation

$$A := B \quad \text{or} \quad B =: A \tag{17}$$

means that A is defined by B.

For a self-adjoint operator H in a Hilbert space $\mathcal{H}$, we denote the corresponding resolution of the identity by $E_H(\lambda)$ ($\lambda \in R^1$) that satisfies

$$\begin{aligned}
&E_H(\lambda)E_H(\mu) = E_H(\min(\lambda,\mu)),\\
&\text{s-}\lim_{\lambda\to-\infty} E_H(\lambda) = 0, \quad \text{s-}\lim_{\lambda\to\infty} E_H(\lambda) = I,\\
&E_H(\lambda+0) = E_H(\lambda)
\end{aligned}$$

where $E_H(\lambda+0) = \text{s-}\lim_{\mu\downarrow\lambda} E_H(\mu)$. Such a family $\{E_H(\lambda)\}_{\lambda\in R^1}$ is uniquely determined by the relation

$$H = \int_{-\infty}^{\infty} \lambda dE_H(\lambda).$$

An operator valued measure $E_H(B)$ is defined by the relation $E_H((a,b]) = E_H(b) - E_H(a)$ from the resolution $\{E_H(\lambda)\}$ of the identity, and is extended to general Borel sets B as a countably additive measure using the properties above of $E_H(\lambda)$.

Let

$$P(\lambda) = E_H(\lambda) - E_H(\lambda - 0)$$

for $\lambda \in R^1$. $P(\lambda) \neq 0$ if and only if λ is an eigenvalue of H. The eigenspace or pure point spectral subspace $\mathcal{H}_p(H)$ for a selfadjoint operator H in a Hilbert space $\mathcal{H}$ is defined by

$$\mathcal{H}_p(H) = \text{the closed linear hull of } \{f \mid Hf = \lambda f \text{ for some } \lambda \in R^1\}.$$

The orthogonal projection P_H onto $\mathcal{H}_p(H)$ is called eigenprojection for H. The continuous spectral subspace $\mathcal{H}_c(H)$ for H is defined by

$$\mathcal{H}_c(H) = \{f \mid E_H(\lambda)f \text{ is strongly continuous with respect to } \lambda \in R^1\}.$$

Then it is seen that $\mathcal{H}_c(H) = \mathcal{H}_p(H)^\perp$. The absolutely continuous subspace $\mathcal{H}_{ac}(H)$ for H is defined by

$$\begin{aligned}
\mathcal{H}_{ac}(H) = \{f \mid &\text{ The measure } (E_H(\Delta)f, f) = \|E_H(\Delta)f\|^2 \text{ is}\\
&\text{absolutely continuous with respect to Lebesgue measure}\}.
\end{aligned}$$

The singular continuous subspace $\mathcal{H}_{sc}(H)$ is then defined by

$$\mathcal{H}_{sc}(H) = \mathcal{H}_c(H) \ominus \mathcal{H}_{ac}(H).$$

Thus

$$\mathcal{H}_c(H) = \mathcal{H}_{ac}(H) \oplus \mathcal{H}_{sc}(H).$$

The part H_p, H_c, H_{ac}, H_{sc} of H in $\mathcal{H}_p(H)$, $\mathcal{H}_c(H)$, $\mathcal{H}_{ac}(H)$, $\mathcal{H}_{sc}(H)$ are spectrally discontinuous, spectrally continuous, spectrally absolutely continuous and spectrally singular continuous, respectively. The spectra of these operators, $\sigma(H_p), \sigma(H_c), \sigma(H_{ac}), \sigma(H_{sc})$ are called point spectrum, continuous spectrum, absolutely continuous spectrum, and singular continuous spectrum of H, respectively.

Contents

I Local Systems **1**

1 Quantum Mechanical Time Contradicts The Uncertainty Principle **3**

2 Position and Momentum **7**

3 Time **11**
- 3.1 Definition of local time . . . 11
- 3.2 Justification of local time as a notion of time . . . 12
- 3.3 Uncertainty of time . . . 23

4 Local Systems **25**

II Motion Inside a Local System **31**

5 Free Hamiltonian **35**
- 5.1 Spectral representation of free Hamiltonian . . . 35
- 5.2 Spatial asymptotics of the free resolvent . . . 41
- 5.3 Propagation estimates for the free evolution . . . 50

6 Two-Body Hamiltonian **61**
- 6.1 Eigenvalues of a two-body Hamiltonian . . . 61
- 6.2 Wave operators . . . 64
- 6.3 Asymptotic completeness . . . 66

7 Many-Body Hamiltonian **83**
- 7.1 Preliminaries . . . 83
- 7.2 Scattering spaces . . . 88
- 7.3 A partition of unity . . . 90
- 7.4 A decomposition of continuous spectral subspace . . . 95
- 7.5 A characterization of the ranges of wave operators . . . 110

III Observation **119**

8 Principle of General Relativity **121**

9 Observation **125**
9.1 Preliminaries 125
9.2 The first step 128
9.3 The second step 130

IV Conclusions **141**

10 Inconsistency of Mathematics? **143**

11 Stationary Universe **149**

12 Existence of Local Motion **157**
12.1 Gödel's theorem 157
12.2 Local time exists 158
12.3 A refined argument 159

Part I

Local Systems

Chapter 1

Quantum Mechanical Time Contradicts The Uncertainty Principle

In classical Newtonian mechanics, one can define mean velocity v by $v = x/t$ of a scattered particle that starts from the origin at time $t = 0$ and arrives at position x at time t, if we assume that the coordinates of space and time are given in an *a priori* sense. This definition of velocity and hence that of momentum do not produce any problems, which assures that in classical regime there is no problem in the notion of space-time. Also in classical relativistic view, this would be valid insofar as we discuss the motion of a particle in the coordinates of the observer's.

Let us consider quantum mechanical case where the space-time coordinates are given *a priori*. Then the mean velocity of a scattered particle that starts from a point around the origin at time 0 and arrives at a point around x at time t should be defined as $v = x/t$. The longer the time length t is, the more exact this value will be, if the errors of the positions at time 0 and t are the same extent, say $\delta > 0$, for all t. This is a definition of the velocity, so this must hold in exact sense if the definition works at all. Thus

$$\text{we have a precise value of (mean) momentum } p = mv \text{ at a large time } t \tag{1.1}$$

with m being the mass of the particle. Note that the mean momentum approaches the momentum at time t when $t \to \infty$ as the interaction of the particle with other particles vanishes as $t \to \infty$ because the particle we are considering is a scattered one so that it escapes to the infinity as $t \to \infty$.

However in quantum mechanics, the uncertainty principle prohibits the position and momentum from taking exact values simultaneously. For illustration we consider a normalized state ψ such that $\|\psi\| = 1$ in one dimensional case. Then the expectation values of the position and momentum operators $Q = x$ and $P = \frac{\hbar}{i}\frac{d}{dx}$ on the state ψ are given by

$$q = (Q\psi, \psi), \quad p = (P\psi, \psi)$$

respectively, and these operators satisfy commutation relation:

$$[P, Q] = PQ - QP = \frac{\hbar}{i}.$$

Further their variances are

$$\Delta q = \|(Q - q)\psi\|, \quad \Delta p = \|(P - p)\psi\|.$$

Thus their product satisfies the inequality

$$\begin{aligned}
\Delta q \cdot \Delta p &= \|(Q - q)\psi\|\|(P - p)\psi\| \geq |((Q - q)\psi, (P - p)\psi)| \\
&= |(Q\psi, P\psi) - qp| \geq |\mathrm{Im}((Q\psi, P\psi) - qp)| \\
&= |\mathrm{Im}(Q\psi, P\psi)| = \left|\frac{1}{2}((PQ - QP)\psi, \psi)\right| \\
&= \left|\frac{1}{2}\frac{\hbar}{i}\right| = \frac{\hbar}{2}.
\end{aligned}$$

Namely

$$\Delta q \cdot \Delta p \geq \frac{\hbar}{2}. \tag{1.2}$$

This uncertainty principle means that there is a least value $\hbar/2(> 0)$ for the product of the variances of position and momentum so that the independence between position and momentum is assured in an absolute sense that there is no way to let position and momentum correlate exactly as in classical views.

Applying (1.2) to the above case of the particle that starts from the origin at time $t = 0$ and arrives at x at time t, we have at time t

$$\Delta p > \frac{\hbar}{2\delta} \tag{1.3}$$

because we have assumed the error Δq of the coordinate x of the particle at time t is less than $\delta > 0$. But the argument (1.1) above tells that $\Delta p \to 0$ when $t \to \infty$, contradicting (1.3).

This observation shows that, if given a pair of *a priori* space and time coordinates, quantum mechanics becomes contradictory.

A possible solution to this apparent contradiction of the notion of space-time in quantum mechanics would be to regard the independent quantities, space and momentum operators, as the fundamental quantities of quantum mechanics, and disregard the notion of time from the framework of quantum mechanics. As time t can be introduced as a ratio x/v on the basis of the notion of space and momentum in this view (see Definition 3.1), time is a redundant notion that should not be given a role independent of space and momentum.

It might be thought that in this view we lose the relation $v = x/t$ that is necessary for the notion of time to be valid, if space and momentum operators are independent as we have seen. However there can be found a relation like $x/t = v$ as an approximate relation that holds to the extent that the relation does not contradict the uncertainty principle (see Theorem 3.2 below).

As a consequence of the abandonment of the *a priori* given time in quantum mechanics, the quantum jump (or wave function collapse) that is assumed to occur in usual quantum mechanics whenever the particles are observed becomes unnecessary in our formulation.

This misconception of quantum jump comes from our unconscious inheritance of the classical view of time to quantum mechanics that the motion is governed by time, and hence the system must evolve along with this given time coordinate. This unconscious assumption urges us to think we observe a definite eigenstate that has sharp values of the quantity that we observe and jumps or collapses must occur when we make observation at one moment in time coordinate. However, what one is able to observe actually is not the eigenstates. No stable eigenstates can be observed as eigenstates, as will be seen in chapter 4, (4.3). Even if we can observe eigenstates, they are necessarily destroyed and become unstable scattered states. We thus observe just the scattering states or processes. We define time as the evolution of these scattering states. Then no eigenstates need appear in the formulation of quantum mechanics. Jumps and eigenstates are ghosts arising from our customary thought that we are accustomed to based on the passed classical notion of time that has been assumed given *a priori*. In more exact words, the usual quantum mechanical theory is an overdetermined system that involves too many independent variables: space, momentum, and time. In that framework time is inevitably not free from the classical image that velocity is defined by $v = x/t$, thus yielding a contradiction discussed above. What is responsible for this misunderstanding is our lack of recognition that the revolution by quantum mechanics of our common sense is too far to be caught by our conventional understanding of the world.

Chapter 2

Position and Momentum

We consider N ($N \geq 1$) particles, which are moving in the Euclidean space R^3. We label them as $1, 2, \cdots, N$.

Let $X_j = (X_{j1}, X_{j2}, X_{j3})$ and $P_j = (P_{j1}, P_{j2}, P_{j3})$ ($j = 1, 2, \cdots, N$) denote the position and momentum operators of the j-th particle. Namely X_{jk} ($k = 1, 2, 3$) is a multiplication operator in $L^2(R^{3N})$ defined by

$$(X_{jk}f)(x) = x_{jk}f(x), \quad (x = (x_{11}, x_{12}, x_{13}, x_{21}, \cdots, x_{N1}, x_{N2}, x_{N3}) \in R^{3N})$$

and P_j is the differential operator

$$(P_j f)(x) = \hbar D_{x_j} f(x) = \frac{\hbar}{i}\frac{\partial f}{\partial x_j}(x) := \frac{\hbar}{i}\left(\frac{\partial f}{\partial x_{j1}}(x), \frac{\partial f}{\partial x_{j2}}(x), \frac{\partial f}{\partial x_{j3}}(x)\right).$$

Here $\hbar = \frac{h}{2\pi}$, where h is the Planck's constant. Their domains are

$$D(X_{jk}) = \{f | f \in L^2(R^{3N}), x_{jk}f(x) \in L^2(R^{3N})\},$$
$$D(P_{jk}) = \{f | f \in L^2(R^{3N}), \frac{\partial f}{\partial x_{jk}}(x) \in L^2(R^{3N})\},$$

where the differentiation is understood in the distribution sense.

X_{jk} and $P_{j'k'}$ satisfy the canonical commutation relation. We write $[A, B] = AB - BA$ for two operators A and B in $L^2(R^{3N})$.

$$\begin{aligned} &[X_{jk}, X_{j'k'}] = 0, \\ &[P_{jk}, P_{j'k'}] = 0, \\ &[X_{jk}, P_{j'k'}] = i\hbar\delta_{jj'}\delta_{kk'}, \end{aligned} \tag{2.1}$$

where $\delta_{j\ell}$ is Kronecker's delta.

Let $m_j > 0$ be the mass of the j-th particle. The Hamiltonian of the system is defined by

$$H = \sum_{j=1}^{N} \frac{1}{2m_j} P_j^2 + \sum_{1 \leq i < j \leq N} V_{ij}(X_i - X_j), \tag{2.2}$$

where $P_j^2 = \sum_{k=1}^3 P_{jk}^2$ and $V_{ij}(x)$ $(x \in R^3)$ is a real-valued pair potential which describes the interaction between the particles i and j. Since this interaction depends only on the relative position $x_i - x_j \in R^3$ of the particles, we can remove the center of mass from the Hamiltonian. Namely, denoting the old variables x_i by X_i with some abuse of notation, we introduce new variables x_i as follows. We first define the center of mass of the N-particle system by

$$X_C = \frac{m_1 X_1 + \cdots + m_N X_N}{m_1 + \cdots + m_N}, \tag{2.3}$$

and then define x_i as Jacobi coordinates:

$$x_i = X_{i+1} - \frac{m_1 X_1 + \cdots + m_i X_i}{m_1 + \cdots + m_i}, \quad i = 1, 2, \cdots, n = N - 1. \tag{2.4}$$

Accordingly, we define the momentum operators $P_C = (P_{C1}, P_{C2}, P_{C3})$ and $p_i = (p_{i1}, p_{i2}, p_{i3})$:

$$P_C = \frac{\hbar}{i}\frac{\partial}{\partial X_C}, \quad p_i = \frac{\hbar}{i}\frac{\partial}{\partial x_i}.$$

It is clear that these satisfy the canonical commutation relation. Using these new X_C, P_C, x_i, p_i, we can rewrite H as

$$H = \tilde{H} + H_C, \tag{2.5}$$

where

$$\tilde{H} = \sum_{i=1}^n \frac{1}{2\mu_i} p_i^2 + \sum_{i<j} V_{ij}(x_{ij}),$$
$$H_C = \frac{1}{2\sum_{j=1}^N m_j} P_C^2,$$

with x_{ij} being the expression of $X_i - X_j$ in the new coordinates, and $\mu_i > 0$ is the reduced mass defined by the relation:

$$\frac{1}{\mu_i} = \frac{1}{m_{i+1}} + \frac{1}{m_1 + \cdots + m_i}.$$

The new coordinates give a decomposition $L^2(R^{3N}) = L^2(R^3) \otimes L^2(R^{3n})$ and in this decomposition, H is written as

$$H = H_C \otimes I + I \otimes \tilde{H}.$$

H_C is the well-known Laplacian, and what we are concerned with is the relative motion of the N-particles. Thus we have only to consider $\tilde{H}$ in the Hilbert space $\mathcal{H} = L^2(R^{3n})$. We rewrite this $\tilde{H}$ as H:

$$H = H_0 + V = \sum_{i=1}^n \frac{1}{2\mu_i} p_i^2 + \sum_{i<j} V_{ij}(x_{ij}) = -\sum_{i=1}^n \frac{\hbar^2}{2\mu_i}\Delta_{x_i} + \sum_{i<j} V_{ij}(x_{ij}), \tag{2.6}$$

where

$$\Delta_{x_i} = \sum_{k=1}^{3} \frac{\partial^2}{\partial x_{ik}^2}.$$

This means that we consider the Hamiltonian H in (2.5) restricted to the subspace

$$(m_1 + \cdots + m_N)X_C = m_1 X_1 + \cdots + m_N X_N = 0 \tag{2.7}$$

of R^{3N}. We equip this subspace with the inner product:

$$\langle x, y\rangle = \sum_{i=1}^{n} \mu_i x_i \cdot y_i, \tag{2.8}$$

where $\cdot$ denotes the Euclidean scalar product. With respect to this inner product, the changes of variables between Jacobi coordinates in (2.4) are realized by orthogonal transformations on the space R^{3n} defined by (2.7), while μ_i and x_i depend on the order of the constitution of Jacobi coordinates in (2.4). If we use this inner product, H_0 can be written as:

$$H_0 = \frac{1}{2}\langle v, v\rangle, \tag{2.9}$$

where $v = (v_1, \cdots, v_n) = (\mu_1^{-1}p_1, \cdots, \mu_n^{-1}p_n)$ is the velocity operator.

It is known that H is a selfadjoint operator in $\mathcal{H}$ under suitable decay assumptions on the pair potentials $V_{ij}(x)$ as $|x| \to \infty$. We consider such a situation in the followings, and precise conditions on $V_{ij}(x)$ will be given when necessary.

We summarize the assumptions we made in this chapter as the following two Axioms 2.1 and 2.2.

Axiom 2.1 *Let $n \geq 1$ and F_{n+1} be a finite subset of $\mathbf{N} = \{1, 2, \cdots\}$ with $\sharp(F_{n+1}) = n+1$. Then for any $j \in F_{n+1}$, there are selfadjoint operators $X_j = (X_{j1}, X_{j2}, X_{j3})$ and $P_j = (P_{j1}, P_{j2}, P_{j3})$ in a tensor product $\mathcal{H}^{n+1} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ of $(n+1)$ times of a separable Hilbert space $\mathcal{H}$, and constants $m_j > 0$ such that*

$$[X_{j\ell}, X_{km}] = 0, \quad [P_{j\ell}, P_{km}] = 0, \quad [X_{j\ell}, P_{km}] = i\delta_{jk}\delta_{\ell m},$$

$$\sum_{j \in F_{n+1}} m_j X_j = 0, \qquad \sum_{j \in F_{n+1}} P_j = 0.$$

The Stone-von Neumann theorem and Axiom 2.1 specify the space dimension (see [1], p.452) as 3 dimension. Namely $\mathcal{H}^n$ is represented as and can be identified with $L^2(R^{3n})$ in the following.

Axiom 2.2 *Let $n \geq 0$ and F_N $(N = n+1)$ be a finite subset of $\mathbf{N} = \{1, 2, \cdots\}$ with $\sharp(F_N) = N$. Let $\{F_N^\ell\}_{\ell=0}^\infty$ be the countable totality of such F_N. Then the local Hamiltonian $H_{n\ell}$ $(\ell \geq 0)$ is of the form*

$$H_{n\ell} = H_{n\ell 0} + V_{n\ell}, \quad V_{n\ell} = \sum_{\substack{\alpha=(i,j) \\ 1 \leq i < j < \infty,\ i,j \in F_N^\ell}} V_\alpha(x_\alpha)$$

on $C_0^\infty(R^{3n})$, where $x_\alpha = x_i - x_j$ ($\alpha = (i,j)$) with x_i being the position vector of the i-th particle, and $V_\alpha(x_\alpha)$ is a real-valued measurable function of $x_\alpha \in R^3$ which is $H_{n\ell 0}$-bounded with $H_{n\ell 0}$-bound of $V_{n\ell}$ less than 1. $H_{n\ell 0} = H_{(N-1)\ell 0}$ is the free Hamiltonian of the N-particle system, which has the form

$$-\sum_{\ell=1}^{n}\sum_{k=1}^{3}\frac{\hbar^2}{2\mu_\ell}\frac{\partial^2}{\partial x_{\ell k}^2} \quad \text{with } \mu_\ell > 0 \text{ being reduced mass.}$$

We remark that the subscript ℓ in $H_{n\ell}$ distinguishes different systems with the same number $N = n+1$ of particles.

This Axiom implies that $H_{n\ell} = H_{(N-1)\ell}$ is uniquely extended to a selfadjoint operator bounded from below in $\mathcal{H}^n = \mathcal{H}^{N-1} = L^2(R^{3(N-1)})$ by the Kato-Rellich theorem. We write $\mathcal{H}_{n\ell} = \mathcal{H}^n$ to indicate that the space $\mathcal{H}^n$ is associated with the Hamiltonian $H_{n\ell}$, and use the notation $(H_{n\ell}, \mathcal{H}_{n\ell})$ to make explict this relation. We will call this pair $(H_{n\ell}, \mathcal{H}_{n\ell})$ a local system (see Definition 4.1).

We do not include vector potentials in the Hamiltonian $H_{n\ell}$ of Axiom 2.2, for we take the position that what is elementary is the electronic charge, and the magnetic forces are the consequence of the motions of charges.

Chapter 3

Time

3.1 Definition of local time

In the previous chapter, we introduced position and momentum operators and defined a Hamiltonian of an N-particle system. All analyses of quantum-mechanical theory are done under the basis of these notions.

The reader might have noticed we do not introduce Schrödinger equation at all. In the usual theory of quantum mechanics, Schrödinger equation is one of the basic assumptions of the theory, without which no analysis of motion of quantum-mechanical particles could be done.

The usual theory of quantum mechanics assumes the *a priori* existence of time when it introduces Schrödinger equation. And the motion of particles is analyzed by the use of the equation along that *a priori* given time.

We reverse the order. We first define time of the system under consideration on the basis of the position and momentum operators. Then we introduce the Schrödinger equation by using that notion of time, which is proper to each system of quantum-mechanical particles. Thus our basic notions of the theory are just position and momentum operators that satisfy the canonical commutation relations.

In this sense, we discard the usual notion of space-time, which is assumed as a fundamental basis of any physical theory. Instead we adopt position and momentum as the fundamental basis of quantum theory. Our degree of freedom in describing nature is thus 6 in place of 4 of space-time that the usual theory assumes. This increase of freedom would make us possible to see nature's properties more precisely than the usual physical theory would.

We leave such precise analysis to the future, and return to the usual description of nature by Schrödinger equation. To do so, we first introduce clock and time of an N-particle system whose Hamiltonian is given by H in (2.6).

Since H in (2.6) is selfadjoint under suitable assumptions on the decay rate of pair potentials $V_{ij}(x)$, we can construct the unitary operator

$$\exp(-itH/\hbar) \tag{3.1}$$

for all real numbers $t \in R^1$. We remark that H is defined by (2.6), and hence $\exp(-itH/\hbar)$ is constructed on the basis of the mere notion of position and momentum operators.

Definition 3.1 *We call the unitary group* $\exp(-itH/\hbar)$ *in (3.1) the* (local or proper) clock *of the system that we are considering, and t in the exponent of* $\exp(-itH/\hbar)$ *the* (local) time *of the system whose Hamiltonian H is given by (2.6).*

3.2 Justification of local time as a notion of time

To see that this definition of time coincides with our intuition, we introduce some notion used in many body scattering theory.

Let $b = \{C_1, \cdots, C_k\}$ be a decomposition of the set $\{1, 2, \cdots, N\}$ into k disjoint subsets $C_1, \cdots, C_k$ of $\{1, 2, \cdots, N\}$. If we denote the number of the elements of a set S by $\sharp(S)$ or $|S|$, we can write $k = \sharp(b) = |b|$. Such a b is called a cluster decomposition of $\{1, 2, \cdots, N\}$.

A clustered Jacobi coordinate $x = (x_b, x^b)$ associated with a cluster decomposition $b = \{C_1, \cdots, C_k\}$ is obtained by first choosing a Jacobi coordinate

$$x^{(C_\ell)} = (x_1^{(C_\ell)}, \cdots, x_{\sharp(C_\ell)-1}^{(C_\ell)}) \in R^{3(\sharp(C_\ell)-1)}, \quad (\ell = 1, \cdots, k)$$

for the $\sharp(C_\ell)$ particles in the cluster C_ℓ, and then by choosing an intercluster Jacobi coordinate

$$x_b = (x_1, \cdots, x_{k-1}) \in R^{3(k-1)}$$

for the centers of mass of the k clusters C_ℓ. Then $x^b = (x^{(C_1)}, \cdots, x^{(C_k)}) \in R^{3(N-k)}$ and $x = (x_b, x^b) \in R^{3(N-1)} = R^{3n}$, and the corresponding canonically conjugate momentum operator is

$$\begin{aligned} &p = (p_b, p^b), \quad p_b = (p_1, \cdots, p_{k-1}), \quad p^b = (p^{(C_1)}, \cdots, p^{(C_k)}) \\ &p_i = \frac{\hbar}{i}\frac{\partial}{\partial x_i}, \quad p^{(C_\ell)} = (p_1^{(C_\ell)}, \cdots, p_{\sharp(C_\ell)-1}^{(C_\ell)}), \quad p_i^{(C_\ell)} = \frac{\hbar}{i}\frac{\partial}{\partial x_i^{(C_\ell)}} \end{aligned}$$

Accordingly $\mathcal{H} = L^2(R^{3n})$ is decomposed:

$$\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}^b, \quad \mathcal{H}_b = L^2(R_{x_b}^{3(k-1)}), \quad \mathcal{H}^b = L^2(R_{x^b}^{3(N-k)}).$$

In this coordinates system, H_0 in (2.6) is decomposed:

$$\begin{aligned} H_0 &= T_b + H_0^b, \\ T_b &= -\sum_{\ell=1}^{k-1} \frac{\hbar^2}{2M_\ell}\Delta_{x_\ell}, \\ H_0^b &= -\sum_{\ell=1}^{k}\sum_{i=1}^{\sharp(C_\ell)-1} \frac{\hbar^2}{2\mu_i^{(C_\ell)}}\Delta_{x_i^{(C_\ell)}}, \end{aligned} \tag{3.2}$$

where Δ_{x_ℓ} and $\Delta_{x_i^{(C_\ell)}}$ are 3-dimensional Laplacians and M_ℓ and $\mu_i^{(C_\ell)}$ are the reduced masses. If we introduce the inner product in the space R^{3n} as in (2.8):

$$\begin{aligned} \langle x, y\rangle &= \langle (x_b, x^b), (y_b, y^b)\rangle = \langle x_b, y_b\rangle + \langle x^b, y^b\rangle \\ &= \sum_{\ell=1}^{k-1} M_\ell x_\ell \cdot y_\ell + \sum_{\ell=1}^{k}\sum_{i=1}^{\sharp(C_\ell)-1} \mu_i^{(C_\ell)} x_i^{(C_\ell)} \cdot y_i^{(C_\ell)}, \end{aligned}$$

and velocity operator

$$v = (v_b, v^b) = M^{-1}p = (m_b^{-1}p_b, (\mu^b)^{-1}p^b),$$

where $M = \begin{pmatrix} m_b & 0 \\ 0 & \mu^b \end{pmatrix}$ is the $3n$-dimensional diagonal mass matrix whose diagonals are given by $M_1, \cdots, M_{k-1}, \mu_1^{(C_1)}, \cdots, \mu_{\sharp(C_k)-1}^{(C_k)}$, then H_0 is written as

$$H_0 = \frac{1}{2}\langle v, v\rangle = T_b + H_0^b = \frac{1}{2}\langle v_b, v_b\rangle + \frac{1}{2}\langle v^b, v^b\rangle.$$

We next decompose the sum of pair potentials in (2.6):

$$\sum_{i<j} V_{ij}(x_{ij}) = V_b + I_b,$$

where

$$\begin{aligned} V_b &= \sum_{C_\ell \in b} V_{C_\ell}, \\ V_{C_\ell} &= \sum_{\{i,j\}\subset C_\ell} V_{ij}(x_{ij}), \\ I_b &= \sum_{\forall C_\ell \in b : \{i,j\} \not\subset C_\ell} V_{ij}(x_{ij}). \end{aligned}$$

By definition, V_{C_ℓ} depends only on the variable $x^{(C_\ell)}$ inside the cluster C_ℓ. Similarly, V_b depends only on the variable $x^b = (x^{(C_1)}, \cdots, x^{(C_k)}) \in R^{3(N-\sharp(b))}$, while I_b depends on all components of the variable x.

Then H in (2.6) is decomposed:

$$\begin{aligned} H &= H_b + I_b = T_b \otimes I + I \otimes H^b + I_b, \\ H_b &= H - I_b = T_b \otimes I + I \otimes H^b, \\ H^b &= H_0^b + V_b. \end{aligned} \tag{3.3}$$

We denote by P_b the orthogonal projection onto the pure point spectral subspace (or eigenspace) $\mathcal{H}_p^b = \mathcal{H}_p(H^b)$ for H^b of $\mathcal{H}^b$. We use the same notation P_b for the obvious extention $I \otimes P_b$ to the total space $\mathcal{H}$. For $\sharp(b) = N$, we set $P_b = I$, and for $\sharp(b) = 1$, we write $P_b = P_H = P$. Let $M = 1, 2, \cdots$ and P_b^M denote an M-dimensional partial projection of P_b such that $\text{s-lim}_{M\to\infty} P_b^M = P_b$. We define for an ℓ-dimensional multi-index $M = (M_1, \cdots, M_\ell)$ $(M_j \geq 1)$ and $\ell = 1, \cdots, n = N-1$

$$\widehat{P}_\ell^M = \left(I - \sum_{\sharp(b_\ell)=\ell} P_{b_\ell}^{M_\ell}\right) \cdots \left(I - \sum_{\sharp(b_2)=2} P_{b_2}^{M_2}\right)(I - P^{M_1}). \tag{3.4}$$

(Note that for $\sharp(b) = 1$, $b = \{C\}$ with $C = \{1, 2, \cdots, N\}$. Thus P^{M_1} is an M_1-dimensional partial projection into the eigenspace of H.) We further define for a $\sharp(b)$-dimensional multi-index $M_b = (M_1, \cdots, M_{\sharp(b)-1}, M_{\sharp(b)}) = (\widehat{M}_b, M_{\sharp(b)})$

$$\widetilde{P}_b^{M_b} = P_b^{M_{\sharp(b)}} \widehat{P}_{\sharp(b)-1}^{\widehat{M}_b}, \quad 2 \leq \sharp(b) \leq N. \tag{3.5}$$

Then it is clear that

$$\sum_{2\le\sharp(b)\le N} \widetilde{P}_b^{M_b} = \widehat{P}_1^{M_1} = I - P^{M_1}, \tag{3.6}$$

provided that the component M_j of M_b depends only on the number j but not on b. In the following we use such M_b's only.

Related with those notions, we denote by $\mathcal{H}_c = \mathcal{H}_c(H)$ the orthogonal complement $\mathcal{H}_p(H)^\perp$ of the eigenspace $\mathcal{H}_p = \mathcal{H}_p(H)$ for the total Hamiltonian H. Namely $\mathcal{H}_c(H)$ is the continuous spectral subspace for H. We note that $\mathcal{H}_c(H) = (I - P_a)\mathcal{H}$ for a unique a with $|a| = 1$, and that for $f \in \mathcal{H}$, $(I - P^{M_1})f \to (I - P_a)f \in \mathcal{H}_c(H)$ as $M_1 \to \infty$. We use freely the notations of functional calculus for selfadjoint operators, e.g. $E_H(B)$ is the spectral measure for H as defined in the section of notation.

Let v_b, as above, denote the velocity operator between the clusters in b. It is expressed as $v_b = m_b^{-1}p_b$ for some $3(\sharp(b) - 1)$-dimensional diagonal mass matrix m_b. To see the meaning of our time in Definition 3.1, we prepare the following

Theorem 3.2 ([10]) *Let $N = n+1 \ge 2$ and let H be the Hamiltonian H in (2.6) or (3.3) for an N-body quantum-mechanical system. Assume that $|X^b|P_b^M$ is a bounded operator for any integer $M \ge 1$. Let suitable conditions on the smoothness and the decay rate of the pair potentials $V_{ij}(x_{ij})$ be satisfied: E.g., assume*

$$|V_{ij}(x)| + |x \cdot (\nabla_x V_{ij})(x)| \to 0 \text{ (as } |x| \to \infty).$$

Let $f \in \mathcal{H}$. Then there exist a sequence $t_m \to \pm\infty$ (as $m \to \pm\infty$) and a sequence M_b^m of multi-indices whose components all tend to ∞ as $m \to \pm\infty$ such that for all cluster decompositions b with $2 \le \sharp(b) \le N$, for all $\varphi \in C_0^\infty(R_{x_b}^{3(\sharp(b)-1)})$, $R > 0$, and $\alpha = \{i, j\}$ that is not included in any $C_\ell \in b$,

$$\|\chi_{\{x||x_\alpha|<R\}}\widetilde{P}_b^{M_b^m} e^{-it_mH/\hbar}f\| \to 0 \tag{3.7}$$

$$\|(\varphi(X_b/t_m) - \varphi(v_b))\widetilde{P}_b^{M_b^m} e^{-it_mH/\hbar}f\| \to 0 \tag{3.8}$$

as $m \to \pm\infty$. Here χ_S is the characteristic function of a set S.

Proof [1]*:* Since mass factors and Planck constant in the definition of the Hamiltonian H are unessential, we may assume $\hbar = 1$ and

$$H = H_0 + V, \quad H_0 = \frac{1}{2}D^2 = -\frac{1}{2}\Delta, \quad V = \sum_{i<j} V_{ij}(x_{ij}).$$

where

$$D = \frac{1}{i}\frac{\partial}{\partial x}, \quad x \in R^{3n}.$$

Let $f \in \mathcal{H}$ satisfy $(1 + |X|)^2 f \in \mathcal{H}$ and $f = E_H(B)f$ for some bounded open set B of R^1. Note that such f's are dense in $\mathcal{H}$. We compute, noting (3.6) and writing $\widetilde{P}_b^{M_b} = \widetilde{P}_b$

[1]Proof here follows that of [10].

and $\widetilde{f} = (I - P^{M_1})f$

$$\sum_{2\le\sharp(b)\le N} e^{itH}\left(\frac{X}{t} - D\right)^2 \widetilde{P}_b e^{-itH} f \tag{3.9}$$

$$= e^{itH}\left(\frac{X}{t} - D\right)^2 e^{-itH}\widetilde{f}$$

$$= e^{itH}\left(\frac{X^2}{t^2} - \frac{2A}{t} + D^2\right) e^{-itH}\widetilde{f}$$

$$= \frac{1}{t^2}\left(e^{itH}X^2 e^{-itH}\widetilde{f} - X^2\widetilde{f}\right) - \frac{2}{t}e^{itH}Ae^{-itH}\widetilde{f} + 2e^{itH}H_0e^{-itH}\widetilde{f} + \frac{X^2}{t^2}\widetilde{f}.$$

Here $A = \frac{1}{2}(X\cdot D + D\cdot X)$. The first term on the RHS is equal to

$$\frac{1}{t^2}\int_0^t e^{isH} i[H_0, X^2]e^{-isH}\widetilde{f}ds.$$

By the relation $i[H_0, X^2] = 2A$, (3.9) is equal to

$$\frac{2}{t^2}\left(\int_0^t e^{isH}Ae^{-isH}\widetilde{f}ds - te^{itH}Ae^{-itH}\widetilde{f}\right) + 2e^{itH}H_0e^{-itH}\widetilde{f} + \frac{X^2}{t^2}\widetilde{f}.$$

The formula in the first parentheses equals

$$\int_0^t e^{isH}Ae^{-isH}\widetilde{f}ds - te^{itH}Ae^{-itH}\widetilde{f}$$

$$= \int_0^t \frac{d}{d\tau}\left(\int_0^\tau e^{isH}Ae^{-isH}\widetilde{f}ds - \tau e^{i\tau H}Ae^{-i\tau H}\widetilde{f}\right)d\tau$$

$$= -\int_0^t se^{isH}i[H, A]e^{-isH}\widetilde{f}ds.$$

Here for any b with $2 \le |b| = \sharp(b) \le N$

$$i[H, A] = i[T_b, A] + i[I_b, A] + i[H^b, A] = 2T_b + i[I_b, A] + i[H^b, A^b],$$

where

$$A^b = \frac{1}{2}(X^b\cdot D^b + D^b\cdot X^b).$$

Thus we have

$$\sum_{2\le\sharp(b)\le N} e^{itH}\left(\frac{X}{t} - D\right)^2 \widetilde{P}_b e^{-itH} f \tag{3.10}$$

$$= -\frac{4}{t^2}\sum_{2\le|b|\le N}\int_0^t se^{isH}T_b\widetilde{P}_b e^{-isH}\widetilde{f}ds + 2e^{itH}H_0e^{-itH}\widetilde{f}$$

$$-\frac{2}{t^2}\sum_{2\le|b|\le N}\int_0^t se^{isH}i[I_b, A]\widetilde{P}_b e^{-isH}\widetilde{f}ds$$

$$-\frac{2}{t^2}\sum_{2\le|b|\le N}\int_0^t se^{isH}i[H^b, A^b]\widetilde{P}_b e^{-isH}\widetilde{f}ds + \frac{X^2}{t^2}\widetilde{f}.$$

Lemma 3.3 *Let $B(s)$ be a continuous family of uniformly bounded operators in $\mathcal{H}$ with respect to s in uniform operator topology. Let $B \subset R^1$ be a bounded open set satisfying $E_H(B)\mathcal{H} \subset \mathcal{H}_c(H)$ and let $2 \le |b| \le N$. Then there is a constant $\epsilon_M > 0$ that goes to 0 when M_j's in the multi-indices M_b's tend to ∞ such that as $T \to \infty$*

$$\left\| \frac{1}{T} \int_0^T B(s) F(|x_\alpha| < R) \widetilde{P}_b^{M_b} e^{-isH} E_H(B) ds \right\| \sim_{\epsilon_M} 0 \tag{3.11}$$

for any pair $\alpha = \{i, j\}$ with $\alpha \notin C_\ell$ for all $C_\ell \in b$. Here $\sim_{\epsilon_M}$ means that the norm of the difference of the both sides is smaller than ϵ_M as $T \to \infty$, and $F(|x_\alpha| < R)$ denotes a smooth positive cut off function which is 1 on the set $S = \{x||x_\alpha| < R\} \subset R^{3n}$ and is 0 outside some neighborhood of S.

By this lemma, the third term on the RHS of (3.10) vanishes as $t \to \infty$ within the small error $\epsilon_M > 0$ determined by the values of M_j in the multi-indices $M_b = (M_1, \cdots, M_\ell)$. The last term on the RHS of (3.10) also vanishes by $X^2 f \in \mathcal{H}$. Thus as $t \to \infty$ we have asymptotically

$$\begin{aligned} &\sum_{2 \le \sharp(b) \le N} e^{itH} \left(\frac{X}{t} - D \right)^2 \widetilde{P}_b e^{-itH} f \\ &\sim_{\epsilon_M} -\frac{4}{t^2} \sum_{2 \le |b| \le N} \int_0^t s e^{isH} T_b \widetilde{P}_b e^{-isH} \widetilde{f} ds + 2 e^{itH} H_0 e^{-itH} \widetilde{f} \\ &\quad -\frac{2}{t^2} \sum_{2 \le |b| \le N} \int_0^t s e^{isH} i[H^b, A^b] \widetilde{P}_b e^{-isH} \widetilde{f} ds. \end{aligned} \tag{3.12}$$

Taking the inner product of the last term with $\widetilde{f} = (I - P^{M_1}) f$ and noting by Lemma 3.3 that as $t \to \infty$

$$\sum_{2 \le |b| \le N} \sum_{2 \le |d| \le N, d \ne b} \frac{2}{t^2} \int_0^t s(\widetilde{f}, e^{isH} (\widetilde{P}_d)^* i[H^b, A^b] \widetilde{P}_b e^{-isH} \widetilde{f}) ds \sim_{\epsilon_M} 0, \tag{3.13}$$

we have by (3.6) as $t \to \infty$

$$\begin{aligned} &\frac{2}{t^2} \sum_{2 \le |b| \le N} \int_0^t s(\widetilde{f}, e^{isH} i[H^b, A^b] \widetilde{P}_b e^{-isH} \widetilde{f}) ds \\ &\sim_{\epsilon_M} \frac{2}{t^2} \sum_{2 \le |b| \le N} \int_0^t s(\widetilde{f}, e^{isH} (\widetilde{P}_b)^* i[H^b, A^b] \widetilde{P}_b e^{-isH} \widetilde{f}) ds \\ &\sim \frac{1}{t} \sum_{2 \le |b| \le N} \int_0^t (\widetilde{f}, e^{isH} (\widetilde{P}_b)^* i[H^b, A^b] \widetilde{P}_b e^{-isH} \widetilde{f}) ds, \end{aligned}$$

provided that the limit as $t \to \infty$ of the RHS exists, which we will prove below. Here $\sim$ means $\sim_0$. Letting $t(s) = s - mS$ for $mS \le s < (m+1)S$ for any fixed $S > 0$, we have by Lemma 3.3 and some commutator arguments as $t \to \infty$

$$\sum_{2\le|b|\le N} \frac{1}{t}\int_0^t (\widetilde{f}, e^{isH}(\widetilde{P}_b)^* i[H^b, A^b]\widetilde{P}_b e^{-isH}\widetilde{f})ds$$
$$\sim_{\epsilon_M} \sum_{2\le|b|\le N} \frac{1}{t}\int_0^t (\widetilde{f}, e^{i(s-t(s))H}(\widetilde{P}_b)^* e^{it(s)H_b} i[H^b, A^b]\widetilde{P}_b e^{-isH}\widetilde{f})ds.$$

This can further be reduced and is asymptotically equal to as $t \to \infty$ with an error $\epsilon_M > 0$

$$\sum_{2\le|b|\le N} \frac{1}{t}\int_0^t (\widetilde{f}, e^{i(s-t(s))H}(\widetilde{P}_b)^* e^{it(s)H_b} i[H^b, A^b]e^{-it(s)H_b}\widetilde{P}_b e^{-i(s-t(s))H}\widetilde{f})ds.$$

Noting $s - t(s) = mS$ for $mS \le s < (m+1)S$, we rewrite this for $t = nS$

$$\begin{aligned}
&\frac{1}{nS}\sum_{m=0}^{n-1}\int_0^S (\widetilde{f}, e^{imSH}(\widetilde{P}_b)^* e^{isH_b} i[H^b, A^b]e^{-isH_b}\widetilde{P}_b e^{-imSH}\widetilde{f})ds \qquad (3.14)\\
&= \frac{1}{n}\sum_{m=0}^{n-1}\frac{1}{S}\int_0^S \frac{d}{ds}(\widetilde{f}, e^{imSH}(\widetilde{P}_b)^* e^{isH_b} A^b e^{-isH_b}\widetilde{P}_b e^{-imSH}\widetilde{f})ds\\
&= \frac{1}{n}\sum_{m=0}^{n-1}\frac{1}{S}[(\widetilde{f}, e^{imSH}(\widetilde{P}_b)^* e^{iSH_b} A^b e^{-iSH_b}\widetilde{P}_b e^{-imSH}\widetilde{f}) - (\widetilde{f}, e^{imSH}(\widetilde{P}_b)^* A^b\widetilde{P}_b e^{-imSH}\widetilde{f})].
\end{aligned}$$

Writing $\widetilde{P}_b = \sum_{j=1}^L P_{b,E_j}\widehat{P}_{|b|-1}$ with P_{b,E_j} being the one dimensional eigenprojection of H^b with eigenvalue E_j, we see that the RHS is bounded by

$$\begin{aligned}
\sum_{j=1}^L \frac{1}{n}\sum_{m=0}^{n-1}\frac{1}{S}\big|&(\widetilde{f}, e^{imSH}(\widetilde{P}_b)^* e^{iS(H^b-E_j)} A^b P_{b,E_j}\widehat{P}_{|b|-1}e^{-imSH}\widetilde{f})\\
&-(\widetilde{f}, e^{imSH}(\widetilde{P}_b)^* A^b P_{b,E_j}\widehat{P}_{|b|-1}e^{-imSH}\widetilde{f})\big|.
\end{aligned}$$

This is arbitrarily small when $S > 0$ is fixed sufficiently large, by our assumption $\||X^b|P_{b,E_j}\| < \infty$.

Summarizing, we have proved when $t \to \infty$

$$\begin{aligned}
&\left(\widetilde{f}, \sum_{2\le\sharp(b)\le N} e^{itH}\left(\frac{X}{t} - D\right)^2 \widetilde{P}_b e^{-itH} f\right) \qquad (3.15)\\
&\sim_{4\epsilon_M} -2\left(\widetilde{f}, \sum_{2\le|b|\le N}\left[\frac{2}{t^2}\int_0^t s e^{isH} T_b\widetilde{P}_b e^{-isH}\widetilde{f}ds - e^{itH}T_b\widetilde{P}_b e^{-itH}\widetilde{f}\right]\right)\\
&\quad +2\left(\widetilde{f}, \sum_{2\le|b|\le N} e^{itH}H_0^b\widetilde{P}_b e^{-itH}\widetilde{f}\right),
\end{aligned}$$

where we have used (3.6) and $H_0 = T_b + H_0^b$. The uniform boundedness in t of the operator

$$(1 + |X|^2)^{-1}(H - i)^{-1}e^{itH}\left(\frac{X}{t} - D\right)^2,$$

and that the projections $P_b^{M_{|b|}}$ are of finite dimension yield for large $R > 1$ with an arbitrarily small error $\delta_R > 0$

$$\left(\widetilde{f}, \sum_{2\le\sharp(b)\le N} e^{itH}\left(\frac{X}{t}-D\right)^2 \widetilde{P}_b e^{-itH} f\right) \tag{3.16}$$
$$\approx \left(\widetilde{f}, \sum_{2\le\sharp(b)\le N} e^{itH}\left(\frac{X}{t}-D\right)^2 F(|x^b|<R)\widetilde{P}_b e^{-itH}\widetilde{f}\right).$$

Here

$$\left(\frac{X}{t}-D\right)^2 = \left(\frac{X_b}{t}-D_b\right)^2 + \left(\frac{(X^b)^2}{t^2} - \frac{2A^b}{t} + 2H_0^b\right).$$

Thus the RHS of (3.16) is asymptotically equal to as $t \to \infty$

$$\left(\widetilde{f}, \sum_{2\le\sharp(b)\le N} e^{itH}\left(\frac{X_b}{t}-D_b\right)^2 F(|x^b|<R)\widetilde{P}_b e^{-itH}\widetilde{f}\right) \tag{3.17}$$
$$+2\left(\widetilde{f}, \sum_{2\le\sharp(b)\le N} e^{itH} H_0^b F(|x^b|<R)\widetilde{P}_b e^{-itH}\widetilde{f}\right).$$

Comparing this with (3.15) and removing $F(|x^b|<R)$ with a small error $\delta_R > 0$, we have as $t \to \infty$

$$\left(\widetilde{f}, \sum_{2\le\sharp(b)\le N} e^{itH}\left(\frac{X_b}{t}-D_b\right)^2 \widetilde{P}_b e^{-itH} f\right) \tag{3.18}$$
$$\sim_{4\epsilon_M+2\delta_R} -2\left(\widetilde{f}, \sum_{2\le|b|\le N}\left[\frac{2}{t^2}\int_0^t se^{isH}T_b\widetilde{P}_b e^{-isH}\widetilde{f}ds - e^{itH}T_b\widetilde{P}_b e^{-itH}\widetilde{f}\right]\right).$$

The both sides do not depend on the cut off $F(|x^b|<R)$, thus we can replace $4\epsilon_M + 2\delta_R$ by $4\epsilon_M$.

We set

$$F(t) = \sum_{2\le|b|\le N}\left[\frac{2}{t^2}\int_0^t se^{isH}T_b\widetilde{P}_b e^{-isH}ds - e^{itH}T_b\widetilde{P}_b e^{-itH}\right],$$
$$G(t) = \sum_{2\le|b|\le N}\left[\frac{2}{t^2}\int_0^t se^{isH}(\widetilde{P}_b)^*T_b\widetilde{P}_b e^{-isH}ds - e^{itH}(\widetilde{P}_b)^*T_b\widetilde{P}_b e^{-itH}\right].$$

By Lemma 3.3, we have when $T \to \infty$ for a large fixed $A > 1$

$$\sum_{2\le|b|\le N}\sum_{2\le|d|\le N, d\ne b}\frac{2}{A}\int_T^{T+A}\left[\frac{2}{t^2}\int_0^t se^{isH}(\widetilde{P}_d)^*T_b\widetilde{P}_b e^{-isH}\widetilde{f}ds - e^{itH}(\widetilde{P}_d)^*T_b\widetilde{P}_b e^{-itH}\widetilde{f}\right]dt \sim_{\epsilon_M} 0.$$

Thus by (3.6), the time mean of the RHS of (3.18) is equal to

$$-\frac{2}{A}\int_T^{T+A}\left(\widetilde{f},F(t)\widetilde{f}\right)dt \sim_{\epsilon_M} -\frac{2}{A}\int_T^{T+A}(\widetilde{f},G(t)\widetilde{f})dt$$

asymptotically as $T\to\infty$. Since the function $H(t)=\frac{2}{t^2}\int_0^t s(\widetilde{f},e^{isH}(\widetilde{P}_b)^*T_b\widetilde{P}_be^{-isH}\widetilde{f})ds$ is real valued, continuously differentiable, uniformly bounded, and its derivative with respect to t goes to 0 as $t\to\infty$, we can find a sequence $T_k\to\infty$ as $k\to\infty$ for each fixed $A>1$ such that the RHS of the above formula goes to 0 as $T=T_k\to\infty$ (see Lemma 8.15 in [9]):

$$\begin{aligned}
&-\lim_{k\to\infty}\frac{2}{A}\int_{T_k}^{T_k+A}(\widetilde{f},G(t)\widetilde{f})dt\\
&=-\lim_{k\to\infty}\frac{2}{A}\int_{T_k}^{T_k+A}\left[\frac{2}{t^2}\int_0^t s(\widetilde{f},e^{isH}(\widetilde{P}_b)^*T_b\widetilde{P}_be^{-isH}\widetilde{f})ds-(\widetilde{f},e^{itH}(\widetilde{P}_b)^*T_b\widetilde{P}_be^{-itH}\widetilde{f})\right]dt\\
&=\lim_{k\to\infty}\frac{1}{A}\int_{T_k}^{T_k+A}t\frac{dH}{dt}(t)dt=0.
\end{aligned}$$

These and (3.18) give when $T_k\to\infty$

$$\frac{1}{A}\int_{T_k}^{T_k+A}\sum_{2\le\sharp(b)\le N}\left(\widetilde{f},e^{itH}\left(\frac{X_b}{t}-D_b\right)^2\widetilde{P}_be^{-itH}f\right)dt\sim_{5\epsilon_M}0.$$

By (3.6) and Lemma 3.3, the LHS is equal to

$$\begin{aligned}
&\frac{1}{A}\int_{T_k}^{T_k+A}\sum_{2\le\sharp(d)\le N}\sum_{2\le\sharp(b)\le N}\left(f,e^{itH}(\widetilde{P}_d)^*\left(\frac{X_b}{t}-D_b\right)^2\widetilde{P}_be^{-itH}f\right)dt\\
&\sim_{\epsilon_M}\frac{1}{A}\int_{T_k}^{T_k+A}\sum_{2\le\sharp(b)\le N}\left\|\left(\frac{X_b}{t}-D_b\right)\widetilde{P}_be^{-itH}f\right\|^2dt
\end{aligned}\tag{3.19}$$

asymptotically as $T_k\to\infty$. Thus we have proved that for given components M_j's of multi-indices M_b's and any fixed large $A>1$

$$\frac{1}{A}\int_{T_k}^{T_k+A}\sum_{2\le\sharp(b)\le N}\left\|\left(\frac{X_b}{t}-D_b\right)\widetilde{P}_be^{-itH}f\right\|^2dt\sim_{6\epsilon_M}0\tag{3.20}$$

as $T_k\to\infty$. By Lemma 3.3, we further have as $T_k\to\infty$

$$\frac{1}{A}\int_{T_k}^{T_k+A}\sum_{2\le\sharp(b)\le N}\left[\left\|\left(\frac{X_b}{t}-D_b\right)\widetilde{P}_be^{-itH}f\right\|^2+\sum_{\forall C_\ell\in b:\alpha\notin C_\ell}\left\|F(|x_\alpha|<R)\widetilde{P}_be^{-itH}f\right\|\right]dt\sim_{7\epsilon_M}0,$$

where we first fix $A>1$ large enough so that the second term is less than ϵ_M and then we let $T_k\to\infty$ (see the proof of Lemma 3.3 below). We can thus take a sequence $\{t_m\}$ tending to ∞ and sequences M_j^m that also tend to ∞ so that

$$\sum_{2\le\sharp(b)\le N}\left\|\left(\frac{X_b}{t_m}-D_b\right)\widetilde{P}_b^{M_b^m}e^{-it_mH}f\right\|^2\to0\tag{3.21}$$

and for all $R > 0$, and $\alpha = \{i, j\}$ that is not in any $C_\ell \in b$

$$\|F(|x_\alpha| < R)\widetilde{P}_b^{M_b^m} e^{-it_m H} f\| \to 0, \tag{3.22}$$

when $m \to \infty$. (3.7) and (3.8) follow from these by density argument. The case $t_m \to -\infty$ is treated similarly.

There remains to prove Lemma 3.3.

Proof of Lemma 3.3: We prove a more general version of Lemma 3.3: Under the assumption of the lemma, we have as $T \to \infty$

$$\left\| \frac{1}{T} \int_0^T B(s) F(|x_\alpha| < R) F(|x^b| < R) \widehat{P}_{|b|-1}^{\widehat{M_b}} e^{-isH} E_H(B) ds \right\| \sim_{\epsilon_M} 0 \tag{3.23}$$

for any $\alpha = \{i, j\}$ such that $\alpha \notin C_\ell$ for all $C_\ell \in b$.

We prove (3.23) by induction on $k = |b|$.

Lemma 3.4 *(3.23) for $|b| = 2$ holds.*

Proof: Since $\|F(|x| > S)F(|x_\alpha| < R)F(|x^b| < R)\| \to 0$ as $S \to \infty$ when $|b| = 2$, $R < \infty$ and $\alpha \notin C_\ell$ for any $C_\ell \in b$ $(\ell = 1, 2)$, we have only to show

$$\lim_{T\to\infty} \left\| \frac{1}{T} \int_0^T B(s) F(|x| < R) E_H(B) e^{-isH} ds \right\| = 0. \tag{3.24}$$

The operator $F(|x| < R)E_H(B)$ is a compact operator. Thus it suffices to prove the lemma with $F(|x| < R)E_H(B)$ replaced by a one dimensional operator $Kf = (f, \phi)\psi$, where $\phi \in \mathcal{H}_c(H)$. Then

$$\begin{aligned} \left\| \frac{1}{T} \int_0^T B(s) K e^{-isH} ds \right\|^2 &= \left\| \frac{1}{T} \int_0^T e^{isH} K^* B(s)^* ds \right\|^2 \\ &= \sup_{\|f\|=1} \left\| \frac{1}{T} \int_0^T e^{isH} K^* B(s)^* f ds \right\|^2 \\ &= \sup_{\|f\|=1} \frac{1}{T^2} \int_0^T \int_0^T (B(s)^* f, \psi)(\psi, B(t)^* f)(e^{-i(t-s)H} \phi, \phi) dt ds \\ &\le C \frac{1}{T^2} \int_0^T \int_0^T |(e^{-i(t-s)H} \phi, \phi)| dt ds \\ &\le C \frac{1}{T} \int_{-T}^T |(e^{-itH} \phi, \phi)| dt \end{aligned} \tag{3.25}$$

for some constant $C > 0$. By Schwarz inequality, the RHS is bounded by

$$\sqrt{2} C \left(\frac{1}{T} \int_{-T}^T |(e^{-itH} \phi, \phi)|^2 dt \right)^{\frac{1}{2}}. \tag{3.26}$$

Noting that the measure $\mu(\lambda) = (E_H(\lambda)\phi, \phi)$ is continuous by $\phi \in \mathcal{H}_c(H)$, we calculate the formula inside the parentheses:

$$\begin{aligned} &\frac{1}{T}\int_{-T}^{T}\int_{R^1}\int_{R^1} e^{-i(\lambda-\lambda')t} d\mu(\lambda)d\mu(\lambda')dt \\ &= 2\int_{R^1}\int_{R^1} \frac{\sin\{(\lambda-\lambda')T\}}{(\lambda-\lambda')T} d\mu(\lambda)d\mu(\lambda'). \end{aligned} \tag{3.27}$$

Dividing the integration region $R^2_{(\lambda,\lambda')}$ into $|\lambda-\lambda'| \le \epsilon$ and the other, we obtain a bound:

$$2\int_{|\lambda-\lambda'|\le\epsilon} d\mu(\lambda)d\mu(\lambda') + \frac{2}{\epsilon T}.$$

The first term can be small arbitrarily if $\epsilon > 0$ is small enough, since the measure $\mu(\lambda)$ is continuous. Then letting $T \to \infty$ we can let the second term go to 0. □

Now assume (3.23) for $|b| < k$ $(3 \le k \le N)$. Let $b = \{C_1, \cdots, C_{|b|}\}$ with $|b| = k$ and assume $\alpha = \{i, j\}$ connects the clusters C_1 and C_2 of b. We denote the new cluster decomposition by $d = \{C_1 \cup C_2, C_3, \cdots, C_k\}$. Then $|d| = k-1$, and $K_1 = F(|x_\alpha| < R)F(|x^b| < R)$ bounds the variable x^d. We decompose $\widehat{P}^{\widehat{M}_b}_{k-1}$ (see (3.4)) in (3.23) as

$$\widehat{P}^{\widehat{M}_b}_{k-1} = (I - P_d^{M_{k-1}})\widehat{P}^{\widehat{M}_d}_{k-2} - \sum_{b_{k-1}\ne d} P^{M_{k-1}}_{b_{k-1}} \widehat{P}^{\widehat{M}_d}_{k-2}, \tag{3.28}$$

where $\widehat{M}_d = (M_1, \cdots, M_{k-2})$. Each $P^{M_{k-1}}_{b_{k-1}}$ on the second term bounds the variable $x^{b_{k-1}}$ with $|b_{k-1}| = k-1$. Since $b_{k-1} \ne d$ and $F(|x_\alpha| < R)F(|x^b| < R)$ bounds the variable x^d, $F(|x_\alpha| < R)F(|x^b| < R)P^{M_{k-1}}_{b_{k-1}}$ connects at least one pair of different two clusters in b_{k-1}. Thus the terms in the second summand on the RHS of (3.28) are treated by the induction hypothesis. Thus we have to show when $T \to \infty$

$$\left\| \frac{1}{T}\int_0^T B(s)F(|x_\alpha| < R)F(|x^b| < R)(I - P_d^{M_{k-1}})\widehat{P}^{\widehat{M}_d}_{k-2} e^{-isH}E_H(B)ds \right\| \sim_{\epsilon_M} 0. \tag{3.29}$$

Let $S > 0$ be arbitrary but fixed and let as before $t(s) = s - mS$ for $mS \le s < (m+1)S$. The norm of (3.29) is bounded by

$$\begin{aligned} &\left\| \frac{1}{T}\int_0^T B(s)K_1(I - P_d^{M_{k-1}})e^{-it(s)H_d}e^{it(s)H}\widehat{P}^{\widehat{M}_d}_{k-2} e^{-isH}E_H(B)ds \right\| \\ &+ \left\| \frac{1}{T}\int_0^T B(s)K_2(I - e^{-it(s)H_d}e^{it(s)H})\widehat{P}^{\widehat{M}_d}_{k-2} e^{-isH}E_H(B)ds \right\|, \end{aligned} \tag{3.30}$$

where $K_1 = F(|x_\alpha| < R)F(|x^b| < R)$ and $K_2 = K_1(I - P_d^{M_{k-1}})$.

Since $H - H_d = I_d$,

$$I - e^{-it(s)H_d}e^{it(s)H} = \int_0^{t(s)} e^{-i\tau H_d} i(H_d - H)e^{i\tau H} d\tau \quad (0 \le t(s) < S) \tag{3.31}$$

is a sum of the terms, each of which bounds at least one variable x_β with $\beta = \{k, m\}$ connecting two different clusters of d. Noting that $F(|x^d| < CR) \geq K_1 = F(|x_\alpha| < R)F(|x^b| < R)$ holds for some constant $C > 0$, we can treat the second term of (3.30) by induction hypothesis.

The first term in (3.30) is rewritten as

$$\left\| \frac{1}{T} \int_0^T B(s) K_1 (I - P_d^{M_{k-1}}) e^{-it(s)H_d} \widehat{P}_{k-2}^{\widehat{M}_d} e^{-i(s-t(s))H} E_H(B) ds \right\| \tag{3.32}$$

with some remainder terms. These remainder terms come from the commutator of $e^{it(s)H}$ and $\widehat{P}_{k-2}^{\widehat{M}_d}$, and can be treated by induction hypothesis. Since $s - t(s) = mS$, (3.32) is rewritten for $t = nS$

$$\left\| \frac{1}{nS} \sum_{m=0}^{n-1} \int_0^S B(s + mS) K_1 (I - P_d^{M_{k-1}}) e^{-isH_d} \widehat{P}_{k-2}^{\widehat{M}_d} E_H(B) ds e^{-imSH} \right\|. \tag{3.33}$$

Since K_1 bounds x^d, the difference

$$K_1\{(I - P_d^{M_{k-1}}) - (I - P_d)\} = K_1(P_d - P_d^{M_{k-1}}) \tag{3.34}$$

tends to 0 as $M_{k-1} \to \infty$ in operator norm. Thus we can replace $K_1(I - P_d^{M_{k-1}})$ in (3.33) by $K_1(I - P_d)$ with an error ϵ_M. This step yields the error ϵ_M in the lemma. To estimate (3.33), letting $S > 0$ large but fixed, we first get an energy cut off for H^d from $E_H(B)$ by some commutator arguments, whose commutators are treated by induction hypothesis. Then we can apply (3.23) for $|b| = 2$ to (3.33) with H replaced by H^d. The proof of (3.23) and Theorem 3.2 is complete. □

We now turn to our purpose of stating Theorem 3.2. I.e., let us see how our definition 3.1 of time coincides with the usual notion of time. (3.8) of Theorem 3.2 or more intuitive form (3.21): for all b with $2 \leq |b| \leq N$

$$\left\| \left(\frac{x_b}{t_m} - v_b \right) \widetilde{P}_b^{M_b^m} e^{-it_m H/\hbar} f \right\| \to 0 \quad (m \to \infty) \tag{3.35}$$

means that the ratio of the position vector x_b and the velocity vector v_b are proportional to time we have defined in Definition 3.1. Namely we have at least schematically

$$\frac{|x_b|}{|v_b|} \sim t \tag{3.36}$$

as t tends to $\pm\infty$ along some sequence $t = t_m$. In this sense, time t of the system we are considering is determined independently of the cluster decomposition b and of the particles inside the system. Thus t has a usual sense of time as a *common* parameter of motion of the system in accordance with the notion of 'common time' in Newton's sense: "relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, $\cdots$" (I. Newton [40] p.6).

Once we have defined time in this way that coincides with our intuition, the motion of the particles is described in the sense of (3.36) by the evolution $\exp(-itH/\hbar)f$ for an initial value wave function f at time $t = 0$. In this sense, we can say that the motion of the particles inside the system obeys the identical equality

$$\left(\frac{\hbar}{i}\frac{d}{dt} + H\right) \exp(-itH/\hbar)f = 0. \tag{3.37}$$

This is the usual Schrödinger equation. Thus in the context where time is already defined as in Definition 3.1, we can say that the motion is governed by Schrödinger equation as usual quantum mechanics assumes as an axiom.

3.3 Uncertainty of time

We have however to remind that this view to nature is *only* an approximation. Time t of the system having Hamiltonian H in (2.6) satisfies the asymptotic relation (3.8) or (3.35), but this is an asymptotic relation and does not give any exact relation like $x_b = tv_b$ which is assumed to hold in classical context. In quantum mechanical context, as we have seen in chapter 1 there is known the uncertainty principle that yields that the values of position and momentum are not determined in precise sense simultaneously, which has been known as a discrepancy between classical and quantum mechanics.

In our context, this uncertainty is understood as that of time in the following sense. Position and momentum are fundamentally independent quantities that satisfy canonical commutation relation (2.1) given in chapter 2. This canonical commutation relation gives the uncertainty relation (see chapter 1) by usual commutation arguments about their variances from the expectation values:

$$\Delta q \cdot \Delta p \geq \frac{\hbar}{2}.$$

This uncertainty means that position and momentum are independent in the sense that there is no way to let position and momentum correlate exactly as in classical views. In the usual formulation where time is given *a priori* as a fundamental quantity, it is anticipated as in chapter 1 that the position x and mean velocity v of a particle are related by the relation $x = tv$ if the particle starts from the origin at time $t = 0$. In this ordinary formulation the uncertainty principle is a cause of a question why position and momentum do not relate exactly to each other. However, in our formulation where we do not introduce *a priori* time t as a fundamental quantity but derive it from position and momentum operators as in Definition 3.1, we need not make any anticipation about their relation other than the ones that follow from (3.8) or its schematic expression (3.36). The uncertainty principle works so as to prohibit the relation $x_b = tv_b$ from being an exact relation and (3.8) gives the most possible extent so that the relation $x_b = tv_b$ holds *without contradicting* the uncertainty principle (for a more precise estimate than (3.8), see e.g., [6]). Thus our local time is defined from the outset to have the uncertainty that is allowed by the relation (3.8).

Chapter 4

Local Systems

So far we have considered an N-particle system and postulated the existence of position and momentum operators for that system. On the basis of these notions we defined time of that N-particle system. Thus space-momentum and time constitute the notions proper to each N-particle system. We call such a system *local system*.

Definition 4.1 *A pair $(H_{n\ell}, \mathcal{H}_{n\ell})$ of a Hilbert space $\mathcal{H}_{n\ell} = \mathcal{H}^n = L^2(R^{3(N-1)})$ $(N = n+1)$ and an N-body Hamiltonian $H_{n\ell}$ is called a local system.*

Then local time t of a local system $(H_{n\ell}, \mathcal{H}_{n\ell})$ is defined as in Definition 3.1. We sometimes denote this time as $t_{(H_{n\ell}, \mathcal{H}_{n\ell})}$ indicating the local system under consideration. Here we recall that the label ℓ distinguishes different local systems with the same number $N = n + 1$ of particles, as remarked in Axiom 2.2 of chapter 2.

There are two regions of research related with local systems. One is the investigation of the properties of the motion inside each local system. Another is the relation between local systems.

The former constitutes the same region as the usual quantum mechanics. One point that is different from the usual view is that the motion is only possible when the initial state f is a scattering state, i.e. when it belongs to the continuous spectral subspace $\mathcal{H}_c(H)$ of the Hamiltonian H, because the usual bound state is unobservable: Let f be a bound state of H with eigenvalue E:

$$Hf = Ef. \tag{4.1}$$

Then its evolution is given by

$$\exp(-itH/\hbar)f(x) = \exp(-itE/\hbar)f(x). \tag{4.2}$$

Therefore, such a state does never change in the sense that the probability density of the existence in the configuration space given by

$$|\exp(-itH/\hbar)f(x)|^2 = |\exp(-itE/\hbar)f(x)|^2 = |f(x)|^2 \tag{4.3}$$

is a constant of motion. Thus such a state cannot be observed except for the case that the state f changes to another state g, by some disturbance from the outside, that involves a part belonging to the continuous spectral subspace of H [2]. In this case what can change in accordance with time is the scattering part of g only. Thus the investigation of quantum-mechanical motion is that of scattering states. The bound states that are considered in usual textbooks should be understood as a certain kind of resonances, which are close to bound states. They are never pure bound states. Thus our purpose in the next part II is to study the evolution $\exp(-itH/\hbar)f$ of scattering states f of an N-body Hamiltonian H in a fixed Hilbert space $\mathcal{H} = L^2(R^{3n})$, where $n = N - 1 \geq 1$.

After some excursion of studying the motion inside a local system, we turn to the investigation of the relation between plural number of local systems in part III. This subject is related with observation. In the explanation of observation, we regard the center of mass of each local system a classical particle behaving under the general principle of relativity and the principle of equivalence. Namely, we postulate that, when one observes other local system than oneself's, the observer observes the motion of the inside of the other local system as following the general theory of relativity, and that the inside of an observed local system is divided in accordance with the purpose of each particular observation. Those subsystems resulting from the division are regarded as classical particles identified with their centers of mass. This postulate would be justified if we see that the observation necessarily identifies the objects of the observation and as a consequence the observer decomposes the observed local system into some number of sublocal systems. Then the observer's investigation would be the relative motion of those sublocal systems. Our postulate here is, therefore, the identification of the center of mass of each subsystem with a classical particle that behaves following classical general relativity. This is our basic assumption on observation.

The problem that the reader might propose at this stage would be why the quantum mechanics which is assumed to have Euclidean space-time structure in the previous parts can be consistent with the postulates of general relativity. This crucial point will be answered at the beginning of part III. The main point is that the space-momentum or space-time structures of different local systems are mutually independent, and hence we can postulate any laws to the relative motion between local systems. We choose the general theory of relativity as our laws that govern the relative motion of local systems, because the theory is known to give sufficiently precise predictions matching the observations or experiments.

We remark that it is possible to interpret special theory of relativity as assuming this kind of setting. Namely, in special theory of relativity, the space-time *inside* a "stationary system[3] " can be interpreted as Euclidean, while the relation between two stationary systems moving with *non-zero* relative speed follows Lorentz transformation and hence in this case the two systems have Lorentz metrics. When we consider inside a stationary system, the Lorentz transformation inside itself can be interpreted as Galilei transformation, because the relative speed with respect to itself is zero. Thus, we arrive at an interpretation of special theory of relativity that the space-time is Euclidean inside a stationary system, while Minkowskian space-time *only* appears when one considers observation of a

[2] I quote the following passage from p. 667 of Paul Busch and Pekka J. Lahti [4]: "In fact, defining (preparing) a physical system in a pure state implies that it is isolated from its environment. Therefore, strictly speaking, it cannot be observed, since an observation entails an interaction which amounts to suspending the system's isolation."

[3] We here use this wording to mean an inertial frame of reference, following p. 38 of A. Einstein [7].

stationary system moving with non-zero velocity with respect to observer's system. The main purpose of the special theory of relativity was to explain the phenomena that occur when making observation of a moving systems from another stationary system. Thus it is understandable that the possibility of adopting Euclidean geometry inside a stationary system has been overlooked for a long time. If one would want to get a unified theory including the zero relative velocity, it is quite natural to regard the Galilei transformation as a special case of Lorentz transformation with zero relative velocity. Therefore the consistency of the Euclidean space-time inside a local system and the outer curved space-time is already inherent in the formulation of special theory of relativity.

The next problem after we show the consistency of the quantum-mechanical Euclidean structure and the general relativistic Riemannian structure of space-time, is the explanation of the results of actual observations or experiments. This we will do to the extent we reach at the present stage of our theory in the remainder of part III. The main ingredients in these explanations are the results we obtain in part II, where normal quantum-mechanical motions are analyzed. Explanation of observation will be done in the same way as in special theory of relativity by assuming that the actually observed values are the ones that are obtained by modifying the usual results of quantum-mechanical calculation by the general relativistic change of coordinates from the observed system to the observer's system.

Exercise

We assume we are given a 3-dimensional coordinate $x = (x_1, x_2, x_3)$ of 3-dimensional Euclidean space R^3 and a corresponding 3-dimensional momentum operator $p = (p_1, p_2, p_3)$ conjugate to x, i.e. we define

$$p_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j}, \quad j = 1, 2, 3,$$

where $\hbar = h/2\pi$, h being Planck constant. We note in the momentum representation x_j works as a differential operator:

$$\mathcal{F} x_j \mathcal{F}^{-1} = i\hbar \frac{\partial}{\partial p_j}.$$

Here $\mathcal{F}$ is the Fourier transformation defined by

$$\mathcal{F}g(p) = (2\pi\hbar)^{-3/2} \int_{R^3} \exp(-ip \cdot x/\hbar) g(x) dx,$$

where $p \cdot x = \sum_{j=1}^{3} p_j x_j$ is the inner product of the vectors x and p. We assume we are given a time parameter t that takes real values. We then define 3-dimensional time and energy operators $T = (t_1, t_2, t_3)$ and $E = (e_1, e_2, e_3)$ for $t \neq 0$ by

$$t_j = t p_j |p|^{-1},$$
$$e_j = \frac{1}{4t}(|p| x_j + x_j |p|),$$

where

$$|p| = \left(\sum_{j=1}^{3} p_j^2 \right)^{1/2}$$

is the positive square root of a nonnegative operator $\sum_{j=1}^{3} p_j^2 = -\hbar^2 \Delta_x$ with Δ_x being Laplacian with respect to x. We note that these operators t_j and e_j initially defined on the space $\mathcal{D} = \mathcal{F}^{-1} C_0^\infty(R^3 - \{0\})$ can be extended to selfadjoint operators in $L^2(R^3)$, where $C_0^\infty(R^3 - \{0\})$ is the space of C^∞-functions with their supports compact in $R^3 - \{0\}$ and $L^2(R^3)$ is the space of square integrable functions on R^3. Clearly they have dimensions of time and energy, respectively, and for $\pm t > 0$, $\pm T = \pm(t_1, t_2, t_3)$ has the same direction as momentum p and satisfies

$$\pm |T| = \pm \left(\sum_{j=1}^{3} t_j^2 \right)^{1/2} = t.$$

For $\pm t > 0$, $\pm E = \pm(e_1, e_2, e_3)$ has almost the same direction as position x, and when x and p denote position and momentum of a scattering particle with mass m whose evolution is governed by a Hamiltonian H, we have

$$|E| \exp(-itH) g = \left(\sum_{j=1}^{3} e_j^2 \right)^{1/2} \exp(-itH) g \sim \frac{|p|^2}{2m} \exp(-itH) g$$

in $L^2(R^3)$ asymptotically as $t \to \pm\infty$ for a scattering state $g \in L^2(R^3)$, since we have $x_j/t \sim p_j/m$ on $\exp(-itH)g$ as $t \to \pm\infty$ along some sequence $t_k \to \pm\infty$. Thus these operators can be regarded as a 3-dimensional version of quantum mechanical time and energy.

Now show that the following uncertainty relation holds between T and E.

Theorem. Let $f \in \mathcal{D} \subset L^2(R^3)$ with its L^2-norm $\|f\| = \langle f, f\rangle^{1/2} = 1$, where

$$\langle f, g\rangle = \int_{R^3} f(x)\overline{g}(x)dx$$

is the inner product of $L^2(R^3)$ with $\overline{g}(x)$ being the complex conjugate of $g(x)$. Let

$$\begin{aligned} \tilde{t}_j &= \langle t_j f, f\rangle, \quad \tilde{e}_j = \langle e_j f, f\rangle, \\ \tilde{T} &= (\tilde{t}_1, \tilde{t}_2, \tilde{t}_3), \quad \tilde{E} = (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3) \end{aligned}$$

be the expectation values of these operators on the state f. Let the variances of T and E be defined by

$$\Delta T = \|(T - \tilde{T})f\| = \left(\sum_{j=1}^{3} \|(t_j - \tilde{t}_j)f\|^2\right)^{1/2},$$

$$\Delta E = \|(E - \tilde{E})f\| = \left(\sum_{j=1}^{3} \|(e_j - \tilde{e}_j)f\|^2\right)^{1/2}.$$

Then we have the uncertainty relation:

$$\Delta T \Delta E \geq \frac{\hbar}{2}.$$

Part II

Motion Inside a Local System

In this part, we will see how the result of Enss [10] stated in Theorem 3.2 gives a solution of the fundamental problem of scattering theory, which is thought to give a basis of any physical observation in the sense stated in the last paragraph of chapter 1.

Before Enss found the so-called Enss' time dependent method in [8], scattering theory had been investigated mainly by stationary method which utilizes the Laplace transform of the propagator $\exp(-itH/\hbar)$. Enss gave a simple method to prove the fundamental property called "asymptotic completeness" of wave operators in [8] by directly utilizing the propagator $\exp(-itH/\hbar)$. Since then the proof of asymptotic completeness can take an elegant form compared with the stationary method by the predecessors (e.g., [16], [22], and the references therein). Though we have to remark that in the case of many-body scattering, some stronger estimates are necessary to establish the asymptotic completeness, and in this book we leave some of those estimates to the references [6], [45], etc.

Theorem 3.2 of Enss played an important role in the interpretation of our formulation of local time in chapter 3. It would be impressive to see that his method also plays an important role in the main part of scattering theory.

We will return to the stationary theory of the Universe in part IV, where the original idea of stationary method by the predecessors will be revived as an important view to nature. These will show that both stationary and time-dependent methods are fundamental in our understanding of nature. We will see in chapter 11 that time-dependent method is based on an artificial introduction of local time after the introduction of the stationary universe. This could be anticipated from our introduction of the notion of local time in chapter 3.

Chapter 5

Free Hamiltonian

5.1 Spectral representation of free Hamiltonian

In this chapter we assume $\hbar = 1$ and consider the free Hamiltonian:

$$H_0 = -\frac{1}{2}\Delta = -\frac{1}{2}\sum_{j=1}^{m}\frac{\partial^2}{\partial x_j^2} = \frac{1}{2}D_x^2 = \frac{1}{2}\sum_{j=1}^{m} D_{x_j}^2, \tag{5.1}$$

defined for functions on R^m with general dimension $m = 1, 2, \cdots$, where

$$D_x = (D_{x_1}, \cdots, D_{x_m}), \quad D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}.$$

When mass factors and Planck constant are necessary, it is easy to include them as in chapter 2 or 3.

We initially define H_0 for functions f from the space $\mathcal{S}$ of rapidly decreasing functions:

$$H_0 f(x) = -\frac{1}{2}\sum_{j=1}^{m}\frac{\partial^2 f}{\partial x_j^2}(x) \tag{5.2}$$

and extend it to a selfadjoint operator in $\mathcal{H} = L^2(R^m)$ with domain $\mathcal{D}(H_0)$ being the Sobolev space of order 2: $H^2(R^m) = \{f \in L^2(R^m) | \int_{R^m} |(1 + D_x^2)f(x)|^2 dx < \infty\}$. We denote the selfadjoint extension also by H_0.

In the following, we denote by (f, g) the inner product of $\mathcal{H} = L^2(R^m)$ and by $\|f\|$ the norm of $\mathcal{H}$.

Let $\mathcal{F}$ be the Fourier transformation: For $f \in \mathcal{S}$

$$\mathcal{F}f(\xi) = (2\pi)^{-\frac{m}{2}}\int_{R^m} e^{-i\xi x} f(x)dx, \tag{5.3}$$

where $\xi x = \sum_{j=1}^{m}\xi_j x_j$ is the Euclidean scalar product of R^m. Then we extend $\mathcal{F}$ as a unitary transformation from $\mathcal{H}$ onto $\mathcal{H}$:

$$\|\mathcal{F}f\| = \|f\|. \tag{5.4}$$

If we denote by $|\xi|^2/2$ the multiplication operator by the function $|\xi|^2/2$ defined in $L^2(R^m_\xi)$, it is easy to see

$$H_0 = \mathcal{F}^{-1}(|\xi|^2/2)\mathcal{F}. \tag{5.5}$$

From this follows that there is no eigenvalue of H_0: for any $f \in \mathcal{D}(H_0)$

$$\exists\lambda \in R^1 : H_0 f = \lambda f \Rightarrow f = 0.$$

Thus $\mathcal{H}_p(H_0) = \{0\}$, and $\mathcal{H}_c(H_0) = \mathcal{H}_p(H_0)^\perp = \mathcal{H} = L^2(R^m)$.

Letting χ_S denote the characteristic function of a set S, we note that the function $(\chi_{(-\infty,\lambda]}(|\xi|^2/2)g, h)$ $(g, h \in \mathcal{H})$ is of bounded variation with respect to $\lambda \in R^1$, and it vanishes for $\lambda \le 0$. Thus we have the relation for any $g, h \in \mathcal{S}$

$$((|\xi|^2/2)g, h) = \int_0^\infty \lambda d_\lambda(\chi_{(-\infty,\lambda]}(|\xi|^2/2)g, h). \tag{5.6}$$

This relation can be extended to $h \in \mathcal{H}$ and g satisfying

$$\int_0^\infty \lambda^2 d\|\chi_{(-\infty,\lambda]}(|\xi|^2/2)g\|^2 < \infty, \tag{5.7}$$

which is equivalent to

$$g \in \mathcal{F}H^2(R^m). \tag{5.8}$$

We define for $f \in \mathcal{H}$

$$E_0(\lambda)f = \mathcal{F}^{-1}\chi_{(-\infty,\lambda]}(|\xi|^2/2)\mathcal{F}f. \tag{5.9}$$

Then by Plancherel formula, we have from (5.5), (5.6) and (5.8) that

$$(H_0 f, g) = \int_0^\infty \lambda d(E_0(\lambda)f, g)$$

for $f \in \mathcal{D}(H_0) = H^2(R^m)$ and $g \in \mathcal{H}$.

It is easy to see by definition that $E_0(\lambda)$ satisfies

$$E_0(\lambda)E_0(\mu) = E_0(\min(\lambda, \mu)), \tag{5.10}$$
$$\text{s-}\lim_{\lambda\to-\infty} E_0(\lambda) = 0, \quad \text{s-}\lim_{\lambda\to\infty} E_0(\lambda) = I, \tag{5.11}$$
$$E_0(\lambda + 0) = E_0(\lambda) \tag{5.12}$$

where $E_0(\lambda + 0) = \text{s-}\lim_{\mu\downarrow\lambda} E_0(\mu)$.

A family of operators that satisfies conditions (5.10)-(5.12) is called a resolution of the identity. It is known (see, e.g., Chapter XI of [50]) that there is a one-to-one correspondence between the resolutions of the identity $\{E(\lambda)\}$ and selfadjoint operators H defined in a Hilbert space $\mathcal{H}$ by the relation

$$(Hf, g) = \int_{-\infty}^\infty \lambda d(E(\lambda)f, g), \quad f \in \mathcal{D}(H), \quad g \in \mathcal{H}. \tag{5.13}$$

Thus $\{E_0(\lambda)\}$ defined by (5.9) is the resolution of the identity corresponding to the selfadjoint operator H_0. We remark that (5.13) yields the relation for any continuous function $F(\lambda)$

$$(F(H)f, g) = \int_{-\infty}^{\infty} F(\lambda) d(E(\lambda)f, g), \quad f \in \mathcal{D}(F(H)), \quad g \in \mathcal{H}. \tag{5.14}$$

Since for $\lambda > 0$

$$\begin{aligned}(\chi_{(-\infty,\lambda]}(|\xi|^2/2)g, h) &= \int_{|\xi|^2/2 \le \lambda} g(\xi)\overline{h(\xi)} d\xi \\ &= \int_0^{\lambda} \int_{S^{m-1}} g(\sqrt{2\mu}\omega)\overline{h(\sqrt{2\mu}\omega)} d\omega (2\mu)^{(m-2)/2} d\mu,\end{aligned}$$

we have for $g, h \in \mathcal{S}$ and $\lambda > 0$

$$\frac{d}{d\lambda}(\chi_{(-\infty,\lambda]}(|\xi|^2/2)g, h) = (2\lambda)^{(m-2)/2} \int_{S^{m-1}} g(\sqrt{2\lambda}\omega)\overline{h(\sqrt{2\lambda}\omega)} d\omega,$$

where $d\omega$ is the surface element of the $(m-1)$-dimensional unit sphere S^{m-1}. Thus we have for $f, g \in \mathcal{S}$ and $\lambda > 0$

$$\begin{aligned}\frac{d}{d\lambda}(E_0(\lambda)f, g) &= \frac{d}{d\lambda}(\chi_{(-\infty,\lambda]}(|\xi|^2/2)\mathcal{F}f, \mathcal{F}g) \\ &= (2\lambda)^{(m-2)/2} \int_{S^{m-1}} (\mathcal{F}f)(\sqrt{2\lambda}\omega)\overline{(\mathcal{F}g)(\sqrt{2\lambda}\omega)} d\omega.\end{aligned} \tag{5.15}$$

We set for $\lambda > 0$

$$\mathcal{F}(\lambda)f(\omega) = (2\lambda)^{(m-2)/4}(\mathcal{F}f)(\sqrt{2\lambda}\omega). \tag{5.16}$$

We remark that by (5.5) we have

$$\mathcal{F}(\lambda)H_0 f(\omega) = \lambda \mathcal{F}(\lambda) f(\omega), \tag{5.17}$$

thus the adjoint $\mathcal{F}(\lambda)^*$ that is defined below is an *eigenoperator* of H_0 in the sense that it satisfies

$$H_0 \mathcal{F}(\lambda)^* = \lambda \mathcal{F}(\lambda)^*. \tag{5.18}$$

(5.15) is now written as

$$\frac{d}{d\lambda}(E_0(\lambda)f, g) = (\mathcal{F}(\lambda)f, \mathcal{F}(\lambda)g)_{L^2(S^{m-1})}, \tag{5.19}$$

where $(\varphi, \psi)_{L^2(S^{m-1})}$ is the inner product of the Hilbert space $L^2(S^{m-1})$. Here if we let $g(\rho) = (\mathcal{F}f)(\rho\ \cdot) \in L^2(S^{m-1})$ for $\rho > 0$, we have for $\lambda > 0$

$$\begin{aligned}\mathcal{F}(\lambda)f &= (\sqrt{2\lambda})^{(m-2)/2}(\mathcal{F}f)(\sqrt{2\lambda}\ \cdot) \\ &= (2\lambda)^{-1/4}\ (\sqrt{2\lambda})^{(m-1)/2} g(\sqrt{2\lambda}) \\ &= (2\lambda)^{-1/4}(2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{i\sqrt{2\lambda}r} \mathcal{F}_\rho(\phi(\rho)\rho^{(m-1)/2} g)(r) dr,\end{aligned} \tag{5.20}$$

where $\phi \in C_0^\infty((0,\infty))$ such that supp $\phi \subset (0,\infty)$ and $\phi(\sqrt{2\lambda}) = 1$, and $\mathcal{F}_\rho g(r)$ is the Fourier transform of $g(\rho)$ with respect to $\rho \in R^1$. Then $\|\mathcal{F}(\lambda)f\|_{L^2(S^{m-1})}$ is bounded by for $s > 1/2$

$$\begin{aligned}&(2\pi)^{-1/2}(2\lambda)^{-1/4}\left(\int_{-\infty}^{\infty}\langle r\rangle^{-2s}dr\right)^{1/2}\left(\int_{-\infty}^{\infty}\langle r\rangle^{2s}\|\mathcal{F}_\rho(\phi(\rho)\rho^{(m-1)/2}g)(r)\|_{L^2(S^{m-1})}^2dr\right)^{1/2}\\&\le C_s\lambda^{-1/4}\|\phi(\rho)\rho^{(m-1)/2}g\|_{H^s((0,\infty),L^2(S^{m-1}))},\end{aligned} \tag{5.21}$$

where $\langle r\rangle = (1+|r|^2)^{1/2}$, and with $\langle D_\rho\rangle^s = \mathcal{F}_\rho^{-1}\langle r\rangle^s\mathcal{F}_\rho$

$$\begin{aligned}&H^s((0,\infty),L^2(S^{m-1})) = \text{ the completion of } C_0^\infty((0,\infty),L^2(S^{m-1})) \\&\text{with respect to the norm } \|h\| = \left(\int_0^\infty \|\langle D_\rho\rangle^s h(\rho)\|_{L^2(S^{m-1})}^2 d\rho\right)^{1/2}.\end{aligned} \tag{5.22}$$

By a calculation, the RHS of (5.21) is bounded by

$$\begin{aligned}&C_{s\phi}\lambda^{-1/4}\left(\int_0^\infty \|\langle D_\rho\rangle^s g(\rho)\|_{L^2(S^{m-1})}^2\rho^{m-1}d\rho\right)^{1/2}\\&\le C_{s\phi}\lambda^{-1/4}\|f\|_{L_s^2},\end{aligned}$$

where $L_s^2 = L_s^2(R^m)$ is the Hilbert space with inner product

$$(f,g)_{L_s^2} = \int_{R^m}\langle x\rangle^{2s}f(x)\overline{g(x)}dx. \tag{5.23}$$

Thus we have proved the estimate: For any $\delta > 0$ and $s > 1/2$, there is a constant $C_{s\delta} > 0$ such that for $\lambda > \delta$

$$\|\mathcal{F}(\lambda)f\|_{L^2(S^{m-1})} \le C_{s\delta}\lambda^{-1/4}\|f\|_{L_s^2}. \tag{5.24}$$

Further, using the expression (5.20) and estimating the difference $\mathcal{F}(\lambda)f - \mathcal{F}(\mu)f$ similarly to the above, we see that $\mathcal{F}(\lambda)$ is continuous in $\lambda > \delta$:

$$\|\mathcal{F}(\lambda)f - \mathcal{F}(\mu)f\|_{L^2(S^{m-1})} \le C_{s\delta}\epsilon(\lambda,\mu)\|f\|_{L_s^2}, \tag{5.25}$$

where with $0 < \theta < s - 1/2$

$$\epsilon(\lambda,\mu) = \left(\int_{-\infty}^{\infty}|e^{i\sqrt{2\lambda}r} - e^{i\sqrt{2\mu}r}|^2\langle r\rangle^{-2s}dr\right)^{1/2} \le C_{\theta\delta}|\lambda-\mu|^\theta \to 0 \tag{5.26}$$

as $\mu \to \lambda$ with $\lambda, \mu > \delta$.

On the other hand, we have for $\varphi \in L^2(S^{m-1})$ and $g \in L_s^2$

$$\begin{aligned}(\mathcal{F}(\lambda)^*\varphi, g)_{L^2(R^m)} &= (\varphi, \mathcal{F}(\lambda)g)_{L^2(S^{m-1})}\\&= \int_{R^m}(2\lambda)^{(m-2)/4}(2\pi)^{-m/2}\int_{S^{m-1}}e^{i\sqrt{2\lambda}\omega y}\varphi(\omega)d\omega\overline{g(y)}dy,\end{aligned} \tag{5.27}$$

which and (5.24) yield

$$\mathcal{F}(\lambda)^*\varphi(x) = (2\pi)^{-m/2}(2\lambda)^{(m-2)/4}\int_{S^{m-1}} e^{i\sqrt{2\lambda}x\omega}\varphi(\omega)d\omega \tag{5.28}$$

and for any $s > 1/2$ and $\lambda > \delta(> 0)$

$$\|\mathcal{F}(\lambda)^*\varphi\|_{L^2_{-s}(R^m)} \leq C_{s\delta}\lambda^{-1/4}\|\varphi\|_{L^2(S^{m-1})}.$$

By (5.25) and (5.27), we further have for $\lambda, \mu > \delta$

$$\|\mathcal{F}(\lambda)^*\varphi - \mathcal{F}(\mu)^*\varphi\|_{L^2_{-s}} \leq C_{s\delta}\epsilon(\lambda,\mu)\|\varphi\|_{L^2(S^{m-1})}.$$

Combining these estimates with (5.19) and (5.25), we have for $\lambda, \mu > \delta$ and $s > 1/2$

$$\left\|\frac{dE_0}{d\lambda}(\lambda)\right\|_{L^2_s\to L^2_{-s}} \leq C_{s\delta}\lambda^{-1/2},$$

and

$$\left\|\frac{dE_0}{d\lambda}(\lambda) - \frac{dE_0}{d\lambda}(\mu)\right\|_{L^2_s\to L^2_{-s}} \leq C_{s\delta}\epsilon(\lambda,\mu),$$

where $\epsilon(\lambda,\mu) \to 0$ as $\mu \to \lambda$.

We now recall the well-known relation for Poisson integral:

$$\frac{1}{\pi}\int_a^b \frac{\epsilon}{(\lambda-\mu)^2+\epsilon^2}h(\lambda)d\lambda \to \begin{cases} 0 & (\mu < a \text{ or } \mu > b) \\ h(\mu) & (a < \mu < b) \end{cases}$$

as $\epsilon \downarrow 0$ for a continuous function $h(\lambda)$. We apply this relation with setting

$$h(\lambda) = \left(\frac{dE_0}{d\lambda}(\lambda)f, g\right), \quad (f, g \in L^2_s, \quad s > 1/2).$$

Then we have for $0 < a < \mu < b < \infty$

$$\begin{aligned} h(\mu) &= \left(\frac{dE_0}{d\lambda}(\mu)f, g\right) \\ &= \lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int_a^b \frac{\epsilon}{(\lambda-\mu)^2+\epsilon^2}\left(\frac{dE_0}{d\lambda}(\lambda)f, g\right)d\lambda \\ &= \lim_{\epsilon\downarrow 0}\frac{1}{2\pi i}\int_a^b\left(\frac{1}{\lambda-\mu-i\epsilon} - \frac{1}{\lambda-\mu+i\epsilon}\right)\left(\frac{dE_0}{d\lambda}(\lambda)f, g\right)d\lambda. \end{aligned} \tag{5.29}$$

By (5.14), the RHS is equal to

$$\begin{aligned} &\lim_{\epsilon\downarrow 0}\frac{1}{2\pi i}\int_{R^1}\left(\frac{1}{\lambda-\mu-i\epsilon} - \frac{1}{\lambda-\mu+i\epsilon}\right)d(E_0(\lambda)E_0(B)f, g) \\ &= \lim_{\epsilon\downarrow 0}\frac{1}{2\pi i}((R_0(z) - R_0(\bar{z}))E_0(B)f, g), \end{aligned} \tag{5.30}$$

where $B = (a, b)$, $z = \mu + i\epsilon$, and

$$R_0(z) = (H_0 - z)^{-1} \tag{5.31}$$

is a resolvent of H_0. Thus we have an expression of $\frac{dE_0}{d\lambda}(\mu)$:

$$\left(\frac{dE_0}{d\lambda}(\mu)f, g\right) = \lim_{\epsilon\downarrow 0} \frac{1}{2\pi i}((R_0(z) - R_0(\bar{z}))E_0(B)f, g)$$

for $\mu \in B = (a, b)$ $(0 < a < b < \infty)$ and $f, g \in L^2_s$ $(s > 1/2)$.

We consider termwise boundary values of $R_0(z) = R_0(\mu \pm i\epsilon)$ as $\epsilon \downarrow 0$. To do so, we write for $f, g \in \mathcal{S}$ using Fourier transform

$$\begin{aligned}(R_0(\mu + i\epsilon)f, g) &= \int_{R^m} \frac{1}{\xi^2/2 - (\mu + i\epsilon)} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi \\ &= \int_0^\infty (2\lambda)^{(m-2)/2} \frac{1}{(\lambda - \mu) - i\epsilon} \int_{S^{m-1}} \hat{f}(\sqrt{2\lambda}\omega)\overline{\hat{g}(\sqrt{2\lambda}\omega)}d\omega d\lambda,\end{aligned} \tag{5.32}$$

where $\hat{f}$ denotes the Fourier transform of f. Setting

$$h(\lambda) = (2\lambda)^{(m-2)/2} \int_{S^{m-1}} \hat{f}(\sqrt{2\lambda}\omega)\overline{\hat{g}(\sqrt{2\lambda}\omega)}d\omega = (\mathcal{F}(\lambda)f, \mathcal{F}(\lambda)g)_{L^2(S^{m-1})}, \tag{5.33}$$

we rewrite the RHS of (5.32) as

$$\int_{-\mu}^\infty \frac{\lambda}{\lambda^2 + \epsilon^2} h(\lambda + \mu)d\lambda + \pi i \frac{1}{\pi} \int_0^\infty \frac{\epsilon}{(\lambda - \mu)^2 + \epsilon^2} h(\lambda)d\lambda. \tag{5.34}$$

The second integral on the RHS is a Poisson integral, thus the limit as $\epsilon \downarrow 0$ exists and satisfies the estimate by (5.24):

$$\left|\lim_{\epsilon\downarrow 0} \frac{1}{\pi} \int_0^\infty \frac{\epsilon}{(\lambda - \mu)^2 + \epsilon^2} h(\lambda)d\lambda\right| \le C_{s\mu}\|f\|_{L^2_s}\|g\|_{L^2_s} \tag{5.35}$$

for $s > 1/2$, where $C_{s\mu} > 0$ is bounded locally uniformly with respect to $\mu > 0$. The first term on the RHS of (5.34) is written for $\delta > 0$ as

$$\begin{aligned}&\int_{-\mu}^\infty \frac{\lambda}{\lambda^2 + \epsilon^2} h(\lambda + \mu)d\lambda \\ &= \left(\int_{-\mu}^{-\delta} + \int_\delta^\infty\right) \frac{\lambda}{\lambda^2 + \epsilon^2} h(\lambda + \mu)d\lambda + \int_{-\delta}^\delta \frac{\lambda}{\lambda^2 + \epsilon^2} h(\lambda + \mu)d\lambda.\end{aligned} \tag{5.36}$$

The limit as $\epsilon \downarrow 0$ of the first term is equal to

$$\left(\int_{-\mu}^{-\delta} + \int_\delta^\infty\right) \frac{1}{\lambda} h(\lambda + \mu)d\lambda = \int_{G_\delta} \frac{1}{|\xi|^2/2 - \mu} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi,$$

where G_δ is the sum of the regions $|\xi|^2/2 \le \mu - \delta$ and $|\xi|^2/2 \ge \mu + \delta$. This is bounded by

$$\frac{1}{\delta}\|f\|_{L^2}\|g\|_{L^2}. \tag{5.37}$$

The second term on the RHS of (5.36) is equal to

$$\int_0^\delta \frac{\lambda}{\lambda^2+\epsilon^2}(h(\mu+\lambda)-h(\mu-\lambda))d\lambda = \int_0^\delta \frac{\lambda^{1+\theta}}{\lambda^2+\epsilon^2}\frac{h(\mu+\lambda)-h(\mu-\lambda)}{\lambda^\theta}d\lambda,$$

where $0 < \theta < s - 1/2$. Thus by applying (5.25) to the definition (5.33) of $h(\lambda)$, we see that the limit as $\epsilon \downarrow 0$ exists and is bounded by

$$\sup_{0<\lambda<\delta} \frac{|h(\mu+\lambda)-h(\mu-\lambda)|}{\lambda^\theta}\int_0^\delta \lambda^{\theta-1}d\lambda \le C_{s\theta\mu}\|f\|_{L^2_s}\|g\|_{L^2_s}$$

for $s > 1/2$ and $s - 1/2 > \theta > 0$, where $C_{s\theta\mu} > 0$ is bounded when μ moves in a compact subset of $(0,\infty)$. Thus the second term on the RHS of (5.36) is bounded by

$$C_{s\mu}\|f\|_{L^2_s}\|g\|_{L^2_s}$$

for $s > 1/2$. Combining this with (5.37) and (5.35), we obtain the existence of the boundary value $R_0(\mu+i0)f = \lim_{\epsilon\downarrow 0} R_0(\mu+i\epsilon)f$ in $L^2_{-s}(R^m)$ and the estimate

$$\|R_0(\mu+i0)\|_{L^2_s\to L^2_{-s}} \le C$$

locally uniformly in $\mu > 0$, where $s > 1/2$. Similarly, the same estimate holds also for $R_0(\mu - i0)$. Reexamining the arguments above, we also see that $R_0(\mu \pm i0)$ is continuous with respect to $\mu > 0$ in operator norm from $L^2_s(R^m)$ into $L^2_{-s}(R^m)$ for $s > 1/2$.

Summarizing, we have proved:

Theorem 5.1 *The resolution $\{E_0(\lambda)\}$ of the identity for H_0 is expressed for $f, g \in L^2_s(R^m)$ $(s > 1/2)$ and $\lambda > 0$ as:*

$$\begin{aligned}\frac{d}{d\lambda}(E_0(\lambda)f,g) &= \frac{1}{2\pi i}((R_0(\lambda+i0)-R_0(\lambda-i0))f,g) \\ &= (\mathcal{F}(\lambda)f,\mathcal{F}(\lambda)g)_{L^2(S^{m-1})}.\end{aligned} \tag{5.38}$$

Here $\{R_0(\lambda\pm i0)\}_{\lambda>0}$ and $\{\mathcal{F}(\lambda)\}_{\lambda>0}$ are continuous families of bounded operators from $L^2_s(R^m)$ $(s > 1/2)$ into $L^2_{-s}(R^m)$ and $L^2(S^{m-1})$, respectively, defined by for $f \in L^2_s(R^m)$

$$R_0(\lambda\pm i0)f = \lim_{\epsilon\downarrow 0} R_0(\lambda \pm i\epsilon)f, \tag{5.39}$$

$$\mathcal{F}(\lambda)f(\omega) = (2\lambda)^{(m-2)/4}(\mathcal{F}f)(\sqrt{2\lambda}\omega). \tag{5.40}$$

In particular, $R_0(\lambda\pm i0)$ and $\mathcal{F}(\lambda)$ are Hölder continuous of order θ $(0 < \theta < s-1/2)$ locally uniformly with respect to $\lambda > 0$, in the uniform operator topology of $B(L^2_s(R^m), L^2_{-s}(R^m))$ and $B(L^2_s(R^m), L^2(S^{m-1}))$, respectively.

5.2 Spatial asymptotics of the free resolvent

Since H_0 is a selfadjoint operator in $\mathcal{H} = L^2(R^m)$, H_0 generates a unitary group

$$U_0(t) = \exp(-itH_0) = e^{-itH_0} \quad (t \in R^1) \tag{5.41}$$

such that

$$U_0(t)U_0(s) = U_0(t+s) \quad (t, s \in R^1). \tag{5.42}$$

If we move to the momentum space $\widehat{\mathcal{H}} = L^2(R^m_\xi)$ by Fourier transformation, we have for $g \in \widehat{\mathcal{S}} = \mathcal{F}\mathcal{S}$ by (5.5)

$$\widehat{U}_0(t)g(\xi) := (\mathcal{F}U_0(t)\mathcal{F}^{-1}g)(\xi) = e^{-it|\xi|^2/2}g(\xi). \tag{5.43}$$

By Lebesgue's dominated convergence theorem, this implies that $\widehat{U}_0(t)g$ is continuous in $t \in R^1$ as an $L^2(R^m_\xi)$-valued function of $t \in R^1$ for general $g \in \mathcal{H}$.

For $f \in \mathcal{S}$

$$\begin{aligned} e^{-itH_0}f(x) &= U_0(t)f(x) = (\mathcal{F}^{-1}\widehat{U}_0(t)\mathcal{F}f)(x) \\ &= (2\pi)^{-m/2}\int_{R^m} e^{ix\xi}e^{-it|\xi|^2/2}(\mathcal{F}f)(\xi)d\xi \\ &= (2\pi)^{-m}\int_{R^m} e^{ix\xi}e^{-it|\xi|^2/2}\int_{R^m} e^{-i\xi y}f(y)dyd\xi \end{aligned} \tag{5.44}$$

Let $\chi(\xi) \in \widehat{\mathcal{S}}$ such that $0 \le \chi(\xi) \le 1$ and $\chi(0) = 1$, and set for $\epsilon > 0$

$$\chi_\epsilon(\xi) = \chi(\epsilon\xi).$$

Then $\chi_\epsilon \in \widehat{\mathcal{S}}$ for each $\epsilon > 0$ and $\chi_\epsilon(\xi) \to 1$ as $\epsilon \downarrow 0$ for each $\xi \in R^m_\xi$. Thus we can rewrite (5.44) using Fubini's theorem as

$$e^{-itH_0}f(x) = (2\pi)^{-m}\lim_{\epsilon\downarrow 0}\int_{R^m}\int_{R^m} e^{i(x\xi - t|\xi|^2/2 - \xi y)}\chi_\epsilon(\xi)f(y)dyd\xi. \tag{5.45}$$

Obviously this limit does not depend on the choice of $\chi \in \widehat{\mathcal{S}}$. We call this type of the limits of integrals oscillatory integrals, and write

$$e^{-itH_0}f(x) = (2\pi)^{-m}\text{Os-}\int\int_{R^{2m}} e^{i(x\xi - t|\xi|^2/2 - \xi y)}f(y)dyd\xi. \tag{5.46}$$

For the simplicity of notation, we introduce a variable

$$\widehat{\xi} = ((2\pi)^{-1}\xi_1, \cdots, (2\pi)^{-1}\xi_m). \tag{5.47}$$

Then (5.46) can be written as

$$e^{-itH_0}f(x) = \text{Os-}\int\int_{R^{2m}} e^{i(x\xi - t|\xi|^2/2 - \xi y)}f(y)dyd\widehat{\xi}. \tag{5.48}$$

We often drop the integration region R^{2m}, when it is obvious from the context, and write

$$e^{-itH_0}f(x) = \text{Os-}\int\int e^{i(x\xi - t|\xi|^2/2 - \xi y)}f(y)dyd\widehat{\xi}. \tag{5.49}$$

We introduce some notations. We call $\alpha = (\alpha_1, \cdots, \alpha_m)$ with the components α_j being nonnegative integers a multi-index. Then we define

$$\begin{aligned}&D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_m}^{\alpha_m}, \quad x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m},\\ &|\alpha| = \alpha_1 + \cdots + \alpha_m,\\ &\langle D_x \rangle = (1 + D_x^2)^{1/2} = (1 - \Delta_x)^{1/2}.\end{aligned} \tag{5.50}$$

Noting the relation

$$D_y^\alpha (e^{-i\xi y}) = (-1)^{|\alpha|} \xi^\alpha (e^{-i\xi y}), \tag{5.51}$$

and integrating by parts inside the integral (5.45) with respect to y, we obtain for $f \in \mathcal{S}$

$$\begin{aligned} e^{-itH_0} f(x) &= (2\pi)^{-m} \lim_{\epsilon \downarrow 0} \int_{R^m} \int_{R^m} e^{i(x\xi - t|\xi|^2/2 - \xi y)} \chi_\epsilon(\xi) f(y) dy d\xi \\ &= \lim_{\epsilon \downarrow 0} \int \int e^{i(x\xi - t|\xi|^2/2 - \xi y)} \chi_\epsilon(\xi) \langle \xi \rangle^{-2m} (\langle D_y \rangle^{2m} f)(y) dy d\widehat{\xi} \\ &= \int \int e^{i(x\xi - t|\xi|^2/2 - \xi y)} \langle \xi \rangle^{-2m} (\langle D_y \rangle^{2m} f)(y) dy d\widehat{\xi}. \end{aligned}$$

If f does not belong to $\mathcal{S}$ but just satisfies, e.g. the conditions

$$\sup_{y \in R^m} |D_y^\alpha f(y)| < \infty$$

for all multi-indices α, we then define $e^{-itH_0} f$ by

$$e^{-itH_0} f(x) = (2\pi)^{-m} \lim_{\epsilon \downarrow 0} \int_{R^m} \int_{R^m} e^{i(x\xi - t|\xi|^2/2 - \xi y)} \chi_\epsilon(\xi) \chi_\epsilon(y) f(y) dy d\xi,$$

and integrate by parts with respect to y and ξ using the relations (5.51) and

$$D_\xi^\alpha (e^{-i\xi y}) = (-1)^{|\alpha|} y^\alpha (e^{-i\xi y}). \tag{5.52}$$

Then we obtain

$$e^{-itH_0} f(x) = \lim_{\epsilon \downarrow 0} \int \int e^{-i\xi y} \langle D_\xi \rangle^{2m} (e^{i(x\xi - t|\xi|^2/2)} \langle \xi \rangle^{-4m} \chi_\epsilon(\xi)) \langle y \rangle^{-2m} \langle D_y \rangle^{4m} (\chi_\epsilon(y) f(y)) dy d\widehat{\xi}.$$

Noting that $\chi_\epsilon(\xi)$ satisfies

$$|D_\xi^\alpha (\chi_\epsilon(\xi))| = |\epsilon^{|\alpha|} (D_\xi^\alpha \chi)(\epsilon \xi)| \le C_\alpha \epsilon^{|\alpha|}$$

for any multi-index α, we see that the equality

$$e^{-itH_0} f(x) = \int \int e^{-i\xi y} \langle D_\xi \rangle^{2m} (e^{i(x\xi - t|\xi|^2/2)} \langle \xi \rangle^{-4m}) \langle y \rangle^{-2m} (\langle D_y \rangle^{4m} f)(y) dy d\widehat{\xi} \tag{5.53}$$

holds. With some smoothness assumptions on the integrands, we always have in this way an expression of oscillatory integrals that does not contain the damping factors like $\chi_\epsilon(\xi)$

or $\chi_\epsilon(y)$. We will use these techniques in the next section in considering the behavior of e^{-itH_0}.

By (5.14), we have the relation for $\mu \in R^1$ and $\epsilon \neq 0$

$$(R_0(\mu \pm i\epsilon)f, g) = \int_0^\infty (\lambda - \mu \mp i\epsilon)^{-1} d(E_0(\lambda)f, g) \tag{5.54}$$

at least for $f, g \in \mathcal{S}$. Noting the relation for $\epsilon > 0$

$$(\lambda - \mu \mp i\epsilon)^{-1} = i \int_0^{\pm\infty} e^{it(\mu \pm i\epsilon - \lambda)} dt, \tag{5.55}$$

and using Fubini's theorem, we can write

$$\begin{aligned}(R_0(\mu \pm i\epsilon)f, g) &= i \int_0^{\pm\infty} \int_0^\infty e^{it(\mu \pm i\epsilon - \lambda)} d(E_0(\lambda)f, g)dt \\ &= i \int_0^{\pm\infty} (e^{it(\mu \pm i\epsilon - H_0)} f, g)dt.\end{aligned}$$

Since $\|e^{it(\mu \pm i\epsilon - H_0)} f\| \le e^{-\epsilon|t|}\|f\|$ in respective signs, we have from this for $f \in \mathcal{H}$

$$R_0(\mu \pm i\epsilon)f = i \int_0^{\pm\infty} e^{it(\mu \pm i\epsilon - H_0)} f dt, \quad (\epsilon > 0, \mu \in R^1). \tag{5.56}$$

Since $e^{itH_0} f$ ($f \in \mathcal{H}$) is continuous in $\mathcal{H}$ with respect to $t \in R^1$ by the remark after (5.43), the integral can be understood as a Riemann integral.

Using the second line of (5.44), we can rewrite (5.56) for $f \in \mathcal{S}$:

$$R_0(\mu \pm i\epsilon)f(x) = (2\pi)^{-m/2} i \int_0^{\pm\infty} \int e^{i(x\xi - t(\xi^2 - 2\mu)/2)} e^{-\epsilon|t|} \hat{f}(\xi) d\xi dt.$$

where $\hat{f} = \mathcal{F}f$. In what follows, assuming $\mu > 0$, we apply stationary phase method to this integral and derive an asymptotic expansion as $r = |x| \to \infty$.

To do so, we assume $\hat{f} \in C_0^\infty(R_\xi^m - \{0\})$ with supp $\hat{f} \subset \{\xi | \ (0 <)a \le |\xi| \le b(< \infty)\}$, and in the integral make a change of variables

$$x = r\omega, \quad r = |x|, \quad t = rs, \qquad (\omega \in S^{m-1}).$$

For the sake of simplicity, we consider the + case only. The − case can be treated similarly. Then we obtain

$$\begin{aligned}-i(2\pi)^{m/2}(R_0(\mu + i\epsilon)f)(r\omega) &= I(r\omega) = I_{\mu\epsilon}(r\omega) \\ &:= r \int_0^\infty \int_{R^m} e^{ir(\omega\xi - s(\xi^2 - 2\mu)/2)} e^{-\epsilon rs} \hat{f}(\xi) d\xi ds.\end{aligned} \tag{5.57}$$

We set

$$\phi = \phi(\mu, \omega; s, \xi) = \omega\xi - s(\xi^2 - 2\mu)/2.$$

Then

$$\partial_\xi \phi = \omega - s\xi, \quad \partial_s \phi = -\xi^2/2 + \mu,$$

where $\partial_\xi = (\partial/\partial\xi_1, \cdots, \partial/\partial\xi_m)$, etc. The solution of $\partial_\xi\phi = 0$ and $\partial_s\phi = 0$ are given by

$$\xi = \xi_c := \sqrt{2\mu}\omega, \quad s = s_c := \frac{1}{\sqrt{2\mu}}.$$

We first divide the integral $I(r\omega)$ as a sum of the integral near $s = 0$ and the one away from $s = 0$. Let $\varphi(s) \in C_0^\infty(R_s^1)$ with supp $\varphi \subset \{s| \; |s| < \frac{1}{2}\min(s_c, \frac{1}{2b})\}$ and $\varphi(s) = 1$ for $|s| \le \frac{1}{4}\min(s_c, \frac{1}{2b})$, and consider

$$I_0(r\omega) = r\int_0^\infty \int e^{ir\phi} e^{-\epsilon rs} \hat{f}(\xi)\varphi(s) d\xi ds.$$

Noting that on supp φ and supp $\hat{f}$

$$|\partial_\xi\phi| = |\omega - s\xi| \ge 1 - \frac{b}{2b} = \frac{1}{2} > 0$$

and using the relation

$$r^{-\ell}(|\partial_\xi\phi|^{-2} i^{-1}\partial_\xi\phi\cdot\partial_\xi)^\ell e^{ir\phi} = e^{ir\phi},$$

we integrate by parts with respect to ξ inside the integral $I_0(r\omega)$. Then we obtain

$$|I_0(r\omega)| \le C_\ell r^{1-\ell}$$

for all $\ell = 1, 2, \cdots$ with the constant $C_\ell > 0$ depending on ℓ but not on $\epsilon > 0$. Thus in the limit as $r \to \infty$, we have only to consider the integral

$$I_\infty(r\omega) = r\int_0^\infty \int e^{ir\phi} e^{-\epsilon rs}\hat{f}(\xi)(1-\varphi)(s) d\xi ds.$$

We next take a function $\chi_\delta \in C_0^\infty(R^{m+1})$ and $\chi_{\delta r} \in C_0^\infty(R^{m+1})$ for $\delta > 0$, $r > 0$ and $1 > \theta > 0$ such that

$$\chi_\delta(s,\xi) = \begin{cases} 1 & (|(s,\xi)| \le \delta) \\ 0 & (|(s,\xi)| \ge 2\delta) \end{cases}$$

$$\chi_{\delta r}(s,\xi) = \chi_\delta(r^\theta((s,\xi) - (s_c,\xi_c)))$$

and divide the integral $I_\infty(r\omega)$ as

$$I_\infty(r\omega) = I_1(r\omega) + I_2(r\omega),$$

where

$$I_1(r\omega) = r\int_0^\infty \int e^{ir\phi} e^{-\epsilon rs}\hat{f}(\xi)\chi_{\delta r}(s,\xi)(1-\varphi)(s) d\xi ds, \tag{5.58}$$

$$I_2(r\omega) = r\int_0^\infty \int e^{ir\phi} e^{-\epsilon rs}\hat{f}(\xi)(1-\chi_{\delta r})(s,\xi)(1-\varphi)(s) d\xi ds. \tag{5.59}$$

On supp $(1-\chi_{\delta r})(s,\xi)$, we have by an argument using the definition of (s_c,ξ_c)

$$|\partial_\xi \phi| + |\partial_s \phi| \geq \rho r^{-2\theta}$$

for some $\rho = \rho_\delta > 0$. Thus if we define a differential operator P whose transposed operator is

$${}^tP = i^{-1}(|\partial_\xi\phi|^2 + |\partial_s\phi|^2)^{-1}(\partial_{(s,\xi)}\phi \cdot \partial_{(s,\xi)}),$$

then we have

$$r^{-\ell}({}^tP)^\ell e^{ir\phi} = e^{ir\phi}.$$

Using this relation, we integrate by parts in the integral $I_2(r\omega)$ and obtain

$$|I_2(r\omega)| \leq C_\ell r^{1-\ell(1-2\theta)}$$

for any $\ell = 1, 2, \cdots$ and $0 < \theta < 1/2$ uniformly in $\epsilon > 0$. Here to assure the integrability with respect to s in $I_2(r\omega)$, we have used the estimates

$$(|\partial_\xi\phi| + |\partial_s\phi|)^{-1} \leq |\partial_\xi\phi|^{-1} = |\omega - s\xi|^{-1} \leq C|s|^{-1}$$

which holds for large $s > 1$ on supp $\hat{f}(1-\varphi) \subset \{(s,\xi)|\ (0<)a \leq |\xi| \leq b(<\infty),\ s \geq \frac{1}{4}\min(s_c, \frac{1}{2b})\}$. When $s > 0$ is small we can use the bound $r^{2\theta}$ as we have obtained above. Thus we can neglect $I_2(r\omega)$ in the limit $r \to \infty$.

We now evaluate $I_1(r\omega)$. We note that the integration region of the integral $I_1(r\omega)$ is included in the compact set supp $\chi_{\delta r}(s,\xi)$ of R^{m+1}, thus the limit as $\epsilon \downarrow 0$ exists and we can drop the factor $e^{-\epsilon rs}$ from $I_1(r\omega)$:

$$I_1(r\omega) = r\int_0^\infty \int e^{ir\phi}\hat{f}(\xi)\chi_{\delta r}(s,\xi)(1-\varphi)(s)d\xi ds. \tag{5.60}$$

To estimate this, we make a Taylor expansion of $\phi = \phi(s,\xi) = \phi(\mu,\omega;s,\xi)$ around (s_c,ξ_c):

$$\phi(s,\xi) = \phi(s_c,\xi_c) + \partial_{(s,\xi)}\phi(s_c,\xi_c)\begin{pmatrix}\tilde{s}\\ \tilde{\xi}\end{pmatrix} + \frac{1}{2}\left\langle J(s,\xi)\begin{pmatrix}\tilde{s}\\ \tilde{\xi}\end{pmatrix}, \begin{pmatrix}\tilde{s}\\ \tilde{\xi}\end{pmatrix}\right\rangle, \tag{5.61}$$

where $\tilde{s} = s - s_c, \tilde{\xi} = \xi - \xi_c$ and $\langle X, Y\rangle = \sum_{j=1}^{m+1} X_j Y_j$ is a scalar product of $X, Y \in R^{m+1}$. By the definition of (s_c,ξ_c), the second term vanishes. The Hessian matrix $J(s,\xi)$ is given by

$$J(s,\xi) = \begin{pmatrix}\partial_s^2\phi(s,\xi) & \partial_s\partial_\xi\phi(s,\xi)\\ \partial_\xi\partial_s\phi(s,\xi) & \partial_\xi^2\phi(s,\xi)\end{pmatrix} = \begin{pmatrix}0 & -\xi\\ -{}^t\xi & -sI_m\end{pmatrix} \tag{5.62}$$

with I_m being the unit matrix of order m. Since $s_c = \frac{1}{\sqrt{2\mu}} > 0$ and $\xi_c = \sqrt{2\mu}\omega \neq 0$, the matrix $J(s,\xi)$ is non-singular on supp $\chi_{\delta r}$ when $r > 1$ is large enough. Further since $J(s,\xi)$ is real symmetric, we can take an orthogonal matrix $P = P(s,\xi)$ such that ${}^tPJ(s,\xi)P$ is a non-singular diagonal matrix:

$$A = A(s,\xi) := {}^tPJ(s,\xi)P = \begin{pmatrix} \frac{-s+\sqrt{s^2+4\xi^2}}{2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{-s-\sqrt{s^2+4\xi^2}}{2} & 0 & \cdots & 0 \\ 0 & 0 & -s & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -s \end{pmatrix}. \tag{5.63}$$

Since A is diagonal, there is a diagonal matrix $Q = Q(s,\xi)$ such that

$${}^tQAQ = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix} =: \mathcal{E}. \tag{5.64}$$

Then since

$${}^tQ{}^tPJ(s,\xi)PQ = \mathcal{E},$$

we have

$$|\det(PQ)| = |\det J(s,\xi)|^{-1/2}.$$

In partcular if we put

$$P_c = P(s_c,\xi_c), \quad Q_c = Q(s_c,\xi_c),$$

we have

$${}^tQ_c{}^tP_cJ(s_c,\xi_c)P_cQ_c = \mathcal{E}, \quad |\det(P_cQ_c)| = |\det J(s_c,\xi_c)|^{-1/2}.$$

We set

$$(s,\xi)(r,\omega) = (s_c,\xi_c)(r,\omega) + \frac{1}{\sqrt{r}}P_cQ_cy, \quad y \in R^{m+1}. \tag{5.65}$$

If we set

$$\widetilde{\mathcal{E}} = {}^tQ_c{}^tP_cJ(s,\xi)P_cQ_c,$$

we have

$$|\widetilde{\mathcal{E}} - \mathcal{E}| \le C|J(s,\xi) - J(s_c,\xi_c)| \le C(|s-s_c| + |\xi-\xi_c|) \le Cr^{-\theta} \tag{5.66}$$

on supp $\chi_{\delta r}$, and we have

$$\phi(s,\xi) = \phi(s_c,\xi_c) + r^{-1}\frac{1}{2}\langle\widetilde{\mathcal{E}}y, y\rangle.$$

Making a change of variables (5.65) and inserting this relation into the definition (5.60) of $I_1(r\omega)$, we obtain

$$I_1(r\omega) = r^{(1-m)/2} e^{ir\phi(s_c,\xi_c)} |\det J(s_c,\xi_c)|^{-1/2} \int_{R^{m+1}} e^{i\frac{1}{2}\langle\widetilde{\mathcal{E}}y,y\rangle} u(r,y)dy,$$

where

$$u(r,y) = \hat{f}(\xi(r,y))\chi_{\delta r}((s,\xi)(r,y))(1-\varphi)(s(r,y)).$$

Noting (5.65), we have for any multi-index α

$$|\partial_y^\alpha u(r,y)| \le C_\alpha \frac{1}{r^{|\alpha|(1/2-\theta)}}, \tag{5.67}$$

where $C_\alpha > 0$ is a constant independent of r, y. We now consider

$$J(r\omega) = \int_{R^{m+1}} e^{\frac{i}{2}\langle\widetilde{\mathcal{E}}y,y\rangle} u(r,y)dy,$$
$$K(r\omega) = \int_{R^{m+1}} e^{\frac{i}{2}\langle\mathcal{E}y,y\rangle} u(r,y)dy.$$

The difference between the two is

$$\begin{aligned} |J(r\omega) - K(r\omega)| &\le \int_{R^{m+1}} |e^{\frac{i}{2}\langle\widetilde{\mathcal{E}}y,y\rangle} - e^{\frac{i}{2}\langle\mathcal{E}y,y\rangle}||u(r,y)|dy \\ &\le C_\kappa \int_{|y|\le Cr^{1/2-\theta}} |\langle(\widetilde{\mathcal{E}}-\mathcal{E})y,y\rangle|^\kappa dy \end{aligned} \tag{5.68}$$

for an arbitrary $1 \ge \kappa > 0$, where we have used that on supp $\chi_{\delta r}$, $|(s,\xi)-(s_c,\xi_c)| = |P_cQ_cy/\sqrt{r}| \le Cr^{-\theta}$. Using (5.66), we have that (5.68) is bounded by

$$\begin{aligned} &\le C_\kappa r^{(1/2-\theta)(m+1)} r^{-\kappa\theta} r^{2(1/2-\theta)\kappa} \\ &= r^{(1/2-\theta)(m+1)-\kappa(3\theta-1)}. \end{aligned} \tag{5.69}$$

Thus if $1/3 < \theta < 1/2$ is close to $1/2$, we can take κ as follows:

$$1 \ge \kappa > \frac{1/2-\theta}{3\theta-1}(m+1)(>0),$$

and we have the negative power on the RHS of (5.69), thus

$$|J(r\omega) - K(r\omega)| \to 0 \quad \text{as } r \to \infty.$$

Thus we have only to consider $K(r\omega)$ instead of $J(r\omega)$ when considering the asymptotic behavior of $I_1(r\omega)$.

Taking the Fourier transform of the two factors in the integrand of $K(r\omega)$, we have by Plancherel formula

$$\begin{aligned} K(r\omega) &= |\det\mathcal{E}|^{-1/2} e^{\pi i \mathrm{sgn}(\mathcal{E})/4} \int_{R^{m+1}} e^{-\frac{i}{2}\langle\mathcal{E}^{-1}\eta,\eta\rangle} \hat{u}(r,\eta)d\eta \\ &= e^{-(m-1)\pi i/4} \int_{R^{m+1}} e^{-\frac{i}{2}\langle\mathcal{E}\eta,\eta\rangle} \hat{u}(r,\eta)d\eta, \end{aligned} \tag{5.70}$$

where $\mathrm{sgn}(\mathcal{E}) = 1 - m$ is the signature of $\mathcal{E}$ and $\hat{u}(r, \eta)$ is the Fourier transform of $u(r, y)$ with respect to y. Taking the Taylor expansion of the exponential function in the integrand, we have

$$\left| e^{-\frac{i}{2}\langle \mathcal{E}\eta, \eta\rangle} - \sum_{j=0}^{\nu-1} \frac{1}{j!}(-i\langle \mathcal{E}\eta, \eta\rangle/2)^j \right| \le \frac{1}{\nu!}|\langle \mathcal{E}\eta, \eta\rangle/2|^\nu. \tag{5.71}$$

Inserting this into (5.70), we obtain

$$\begin{aligned} \left| K(r\omega) - (2\pi)^{(m+1)/2} e^{-(m-1)\pi i/4} \sum_{|\alpha|<2\nu} c_\alpha D_y^\alpha u(r, 0) \right| &\le C_\nu \sum_{|\beta|=2\nu} \left| \int \eta^\beta \hat{u}(r, \eta) d\eta \right| \quad (5.72) \\ &\le C'_\nu \sum_{2\nu \le |\beta| \le 2\nu+m+2} \int |D_y^\beta u(r, y)| dy, \end{aligned}$$

where

$$c_\alpha = \frac{1}{\alpha!}\, \partial_\eta^\alpha (e^{-\frac{i}{2}\langle \mathcal{E}\eta, \eta\rangle})\Big|_{\eta=0} \tag{5.73}$$

vanishes for odd $|\alpha|$. Since the support of $u(r, y)$ with respect to y is included in the ball of radius $cr^{1/2-\theta}$ with center 0 in R^{m+1}, we obtain the bound of the RHS of (5.72) from (5.67):

$$Cr^{((m+1)-2\nu)(1/2-\theta)}. \tag{5.74}$$

Taking $\nu > (m + 1)/2$, we have an expansion formula for $I_1(r\omega)$. In particular, as the first approximation we get for large $r > 1$

$$\left| I_1(r\omega) - (2\pi)^{(m+1)/2} r^{(1-m)/2} e^{-(m-1)\pi i/4} e^{ir\phi(s_c, \xi_c)} |\det J(s_c, \xi_c)|^{-1/2} \hat{f}(\xi_c) \right| = o(r^{-(m-1)/2}).$$

Noting that

$$\phi(s_c, \xi_c) = \sqrt{2\mu}, \quad |\det J(s_c, \xi_c)| = (2\mu)^{-(m-3)/2},$$

we obtain returning to (5.57)

$$\begin{aligned} &R_0(\mu + i0) f(r\omega) \\ &= \sqrt{2\pi} e^{-(m-3)\pi i/4} (2\mu)^{(m-3)/4} e^{i\sqrt{2\mu}r} r^{-(m-1)/2} (\mathcal{F}f)(\sqrt{2\mu}\omega) + o(r^{-(m-1)/2}) \end{aligned}$$

as $r \to \infty$ for $\hat{f} \in C_0^\infty(R^m - \{0\})$. Similarly we can show

$$\begin{aligned} &R_0(\mu - i0) f(r\omega) \\ &= \sqrt{2\pi} e^{(m-3)\pi i/4} (2\mu)^{(m-3)/4} e^{-i\sqrt{2\mu}r} r^{-(m-1)/2} (\mathcal{F}f)(-\sqrt{2\mu}\omega) + o(r^{-(m-1)/2}) \end{aligned}$$

as $r \to \infty$.

Reversing the order of expression we obtain a relation between the Fourier transform $\mathcal{F}(\mu)$ defined by (5.16) and the spatial asymptotics of the free resolvent $R_0(\mu \pm i0)$:

Theorem 5.2 *For $\mathcal{F}f \in C_0^\infty(R^m - \{0\})$ and $\mu > 0$, $\omega \in S^{m-1}$, one has*

$$\mathcal{F}(\mu) f(\pm\omega) = (2\pi)^{-1/2} e^{\pm(m-3)\pi i/4} (2\mu)^{1/4} \lim_{r\to\infty} r^{(m-1)/2} e^{\mp i\sqrt{2\mu}r} (R_0(\mu \pm i0) f)(r\omega). \tag{5.75}$$

5.3 Propagation estimates for the free evolution

In this section we consider some estimates of e^{-itH_0} which are stronger version of Theorem 3.2 in the free Hamiltonian case. For this purpose we introduce the notion of pseudo-differential operator (we call this a ψdo in the following). P is called a ψdo with symbol $p(x,\xi)$, if it is written for $f \in \mathcal{S}$

$$Pf(x) = \text{Os-}\int_{R^m}\int_{R^m} e^{i(x-y)\xi} p(x,\xi) f(y) dy d\widehat{\xi}, \tag{5.76}$$

where $\widehat{\xi}$ is the variable defined by (5.47). We call $p(x,\xi)$ the symbol of the ψdo P and write $p(x,\xi) = \sigma(P)(x,\xi)$. For (5.76) to be well-defined as an oscillatory integral, we need to assume some smoothness conditions on the symbol $p(x,\xi)$ of P. E.g. let us assume that $p(x,\xi)$ is C^∞ with respect to (x,ξ) and satisfies the estimates

$$\sup_{(x,\xi)\in R^{2m}} |\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| < \infty \tag{5.77}$$

for all multi-indices α and β. Let as before $\chi \in \mathcal{S}(R^m)$ such that $\chi(0) = 1$, and set $\chi_\epsilon(\xi) = \chi(\epsilon\xi)$, and define (5.76) by

$$Pf(x) = \lim_{\epsilon\downarrow 0} \int\int e^{i(x-y)\xi} p(x,\xi)\chi_\epsilon(\xi) f(y) dy d\widehat{\xi}. \tag{5.78}$$

Then using the relation

$$(1 + D_y^2)e^{-iy\xi} = (1 + |\xi|^2)e^{-iy\xi},$$

we integrate by parts inside the integral (5.78):

$$\begin{aligned} Pf(x) &= \lim_{\epsilon\downarrow 0} \int\int e^{i(x-y)\xi} p(x,\xi)\chi_\epsilon(\xi)(1+|\xi|^2)^{-m}(1+D_y^2)^m f(y) dy d\widehat{\xi} \\ &= \int\int e^{i(x-y)\xi} p(x,\xi)(1+|\xi|^2)^{-m}(1+D_y^2)^m f(y) dy d\widehat{\xi}. \end{aligned}$$

Thus Pf is well-defined for $f \in \mathcal{S}$ as an oscillatory integral independently of the choice of the damping factor χ_ϵ. We write Pf as $Pf = p(X, D_x)f$ to indicate the symbol $p(x,\xi)$ used in the definition (5.76).

As other forms of the definition of ψdo, we can adopt

$$Qf(x) = q(D_x, X')f(x) = \text{Os-}\int\int e^{i(x-y)\xi} q(\xi,y) f(y) dy d\widehat{\xi},$$

or

$$Pf(x) = p(X, D_x, X')f(x) = \text{Os-}\int\int e^{i(x-y)\xi} p(x,\xi,y) f(y) dy d\widehat{\xi},$$

where the symbols $q(\xi,y)$ and $p(x,\xi,y)$ satisfy

$$\sup_{(\xi,y)\in R^{2m}} |\partial_\xi^\alpha \partial_y^\beta q(\xi,y)| < \infty,$$

$$\sup_{(x,\xi,y)\in R^{3m}} |\partial_x^\alpha \partial_\xi^\beta \partial_y^\gamma p(x,\xi,y)| < \infty$$

for all multi-indices α, β, γ. The relation between these expressions is given by

Proposition 5.3 *Let $p(x,\xi,y)$ be as above. Then $Pf = p(X,D_x,X')f$ is written as*

$$\begin{aligned} Pf(x) &= p_L(X,D_x)f(x) = \text{Os-}\int\int e^{i(x-y)\xi}p_L(x,\xi)f(y)dyd\widehat{\xi} && (5.79)\\ &= p_R(D_x,X')f(x) = \text{Os-}\int\int e^{i(x-y)\xi}p_R(\xi,y)f(y)dyd\widehat{\xi}, && (5.80)\end{aligned}$$

where $p_L(x,\xi)$ and $p_R(\xi,y)$ are defined by

$$p_L(x,\xi) = \text{Os-}\int\int e^{-iy\eta}p(x,\xi+\eta,x+y)dyd\widehat{\eta}, \tag{5.81}$$

$$p_R(\xi,y) = \text{Os-}\int\int e^{iz\eta}p(y+z,\xi+\eta,z)dzd\widehat{\eta}. \tag{5.82}$$

Proof is easily done by using Fourier transformation, and is left to the reader.

It is easy to see that for $P = p(X,D_x,X')$, the adjoint operator P^* is given by

$$P^*f(x) = \text{Os-}\int\int e^{i(x-y)\xi}\overline{p(y,\xi,x)}f(y)dyd\widehat{\xi}.$$

Thus if we consider the operator P^*P, we have the integral operator

$$P^*Pf(x) = \text{Os-}\int\int e^{i(x-y)\xi}r(x,\xi,y)f(y)dyd\widehat{\xi},$$

where

$$r(x,\xi,y) = \text{Os-}\int\int e^{-iz\eta}\overline{p(x+z,\xi+\eta,x)}p(x+z,\xi,y)dzd\widehat{\eta}. \tag{5.83}$$

If we define the semi-norms for the symbol $p(x,\xi,y)$ by

$$|p|_\ell = \sup_{|\alpha|+|\beta|+|\gamma|\le\ell}\ \sup_{x,\xi,y\in R^m}|\partial_x^\alpha\partial_\xi^\beta\partial_y^\gamma p(x,\xi,y)|$$

for $\ell = 0,1,2,\cdots$, then we have for r above

$$|r|_\ell \le C_\ell^2|p|_{\ell'}^2 \tag{5.84}$$

for some constant $C_\ell > 0$, where

$$\ell' = \ell + 2m_0, \quad m_0 = 2[m/2+1].$$

Here for a real number s, $[s]$ denotes the greatest integer that does not exceed s. This is seen by integrating by parts in (5.83) by using the relations

$$(1+D_z^2)e^{-iz\eta} = (1+|\eta|^2)e^{-iz\eta}, \quad (1+D_\eta^2)e^{-iz\eta} = (1+|z|^2)e^{-iz\eta}, \tag{5.85}$$

and noting that

$$\int_{R^m}\int_{R^m}(1+|\eta|^2)^{-[m/2+1]}(1+|z|^2)^{-[m/2+1]}dzd\eta < \infty. \tag{5.86}$$

For the product of $\nu+1$ ($\nu \geq 1$) ψdo's

$$P_j f(x) = \text{Os-}\int\int e^{i(x-y)\xi} p_j(x,\xi,y) f(y) dy d\widehat{\xi}, \quad (j = 1, 2, \cdots, \nu+1),$$

we rewrite $P_j f$:

$$P_j f(x) = \text{Os-}\int\int e^{-iy\xi} p_j(x,\xi,x+y) f(x+y) dy d\widehat{\xi}$$

and calculate the product $Q_{\nu+1} = P_1 \cdots P_{\nu+1}$. Then we have

$$Q_{\nu+1} f(x) = \text{Os-}\int\int e^{i(x-x')\xi} q_{\nu+1}(x,\xi,x') f(x') dx' d\xi.$$

Here

$$\begin{aligned} &q_{\nu+1}(x,\xi,x') \qquad (5.87) \\ &= \text{Os-}\overbrace{\int\cdots\int}^{2\nu} e^{-i\sum_{j=1}^{\nu} y^j \eta^j} \prod_{j=1}^{\nu} p_j(x+\bar{y}^{j-1}, \xi+\eta^j, x+\bar{y}^j) p_{\nu+1}(x+\bar{y}^\nu, \xi, x') d\boldsymbol{y}^\nu d\widehat{\boldsymbol{\eta}}^\nu, \end{aligned}$$

where

$$\begin{aligned} &\bar{y}^0 = 0, \quad \bar{y}^j = y^1 + \cdots + y^j \ (j = 1, 2, \cdots, \nu), \\ &d\boldsymbol{y}^\nu = dy^1 \cdots dy^\nu, \quad d\widehat{\boldsymbol{\eta}}^\nu = d\widehat{\eta}^1 \cdots d\widehat{\eta}^\nu. \end{aligned}$$

Using (5.85), we integrate by parts in (5.87):

$$\begin{aligned} &q_{\nu+1}(x,\xi,x') \qquad (5.88) \\ &= \text{Os-}\overbrace{\int\cdots\int}^{2\nu} e^{-i\sum_{j=1}^{\nu} y^j \eta^j} \prod_{\ell=1}^{\nu} (1+|y^\ell|^{m_0})^{-1} (1 + D_{\eta^\ell}^{m_0}) \\ &\times \prod_{j=1}^{\nu} p_j(x+\bar{y}^{j-1}, \xi+\eta^j, x+\bar{y}^j) p_{\nu+1}(x+\bar{y}^\nu, \xi, x') d\boldsymbol{y}^\nu d\widehat{\boldsymbol{\eta}}^\nu. \end{aligned}$$

We then make a change of variables

$$z^j = y^1 + \cdots + y^j \quad (j = 1, 2, \cdots, \nu),$$

which is equivalent to

$$y^j = z^j - z^{j-1}, \quad z^0 = 0.$$

Noting

$$\sum_{j=1}^{\nu} y^j \eta^j = \sum_{k=1}^{\nu} z^k (\eta^k - \eta^{k+1}), \quad \eta^{k+1} = 0,$$

we again integrate by parts in (5.88):

$$\begin{aligned} & q_{\nu+1}(x,\xi,x') \qquad (5.89)\\ &= \text{Os-}\overbrace{\int\cdots\int}^{2\nu} e^{-i\sum_{k=1}^{\nu} z^k(\eta^k-\eta^{k+1})} \prod_{k=1}^{\nu}(1+|\eta^k-\eta^{k+1}|^{m_0})^{-1}(1+D_{z_k}^{m_0})\\ &\times \prod_{\ell=1}^{\nu}(1+|z^\ell-z^{\ell-1}|^{m_0})^{-1}(1+D_{\eta^\ell}^{m_0})\\ &\times \prod_{j=1}^{\nu} p_j(x+z^{j-1},\xi+\eta^j,x+z^j)p_{\nu+1}(x+z^\nu,\xi,x')d\boldsymbol{z}^\nu d\widehat{\boldsymbol{\eta}}^\nu. \end{aligned}$$

By (5.86), we now obtain the estimate:

$$|q_{\nu+1}(x,\xi,x')| \le C_0^{\nu+1}\prod_{j=1}^{\nu+1}|p_j|_{3m_0}$$

for some constant $C_0 > 0$ independent of ν. Differentiating the both sides of (5.87) and estimating similarly, we have with ℓ_j being 3-dimensional mulit-indices

$$|q_{\nu+1}(x,\xi,x')|_\ell \le C_0^{\nu+1}\sum_{|\ell_1+\cdots+\ell_{\nu+1}|\le\ell}\ \prod_{j=1}^{\nu+1}|p_j|_{3m_0+|\ell_j|}. \qquad (5.90)$$

Note that the constant C_ℓ in (5.84) depends on ℓ, while in the present estimate (5.90) it is replaced by the sum $\sum_{|\ell_1+\cdots+\ell_{\nu+1}|\le\ell}$, which enables us to take the constant C_0 independent of ℓ.

Using this estimate we prove

Theorem 5.4 *Let $p(x,\xi,y)$ satisfy $|p|_\ell < \infty$ for all $\ell \le 3m_0$. Then $P = p(X, D_x, X')$ defines a bounded operator from $\mathcal{H} = L^2(R^m)$ into itself satisfying*

$$\|P\| \le C_0|p|_{3m_0} \qquad (5.91)$$

for the constant $C_0 > 0$ in (5.90).

Proof[4]*:* Let $\chi \in C_0^\infty(B)$, $\chi(0) = 1$ and $0 \le \chi(x) \le 1$, where $B = \{(x,\xi,x')|\ \max(|x|,|\xi|,|x'|) \le 1\}$, and set for $0 < \epsilon < 1$

$$p_\epsilon(x,\xi,x') = \chi(\epsilon x,\epsilon\xi,\epsilon x')p(x,\xi,x').$$

Set

$$K(x,x') = \int_{R^m} e^{i(x-x')\xi}p_\epsilon(x,\xi,x')d\widehat{\xi}.$$

[4]Proof here follows that of [34] p.224

Then

$$P_\epsilon f(x) = p_\epsilon(X, D_x, X')f(x) = \int_{R^m} K(x,x')f(x')dx'$$

satisfies the estimate for $f \in \mathcal{S}$

$$\|P_\epsilon f\|_{\mathcal{H}} \le V_\epsilon^2 |p|_0 \|f\|_{\mathcal{H}}, \tag{5.92}$$

where $V_\epsilon > 0$ is the volume of the ball $|\xi| < \epsilon^{-1}$ in R^m_ξ.

Now set for $\nu = 2^\ell$ $(\ell = 0, 1, 2, \cdots)$

$$Q_{\epsilon\nu} = \overbrace{(P_\epsilon^* P_\epsilon)\cdots(P_\epsilon^* P_\epsilon)}^{\nu}.$$

Then by $\sigma(P_\epsilon^*)(x,\xi,x') = \overline{p_\epsilon(x',\xi,x)}$ and by (5.87), the support of the symbol $\sigma(Q_{\epsilon\nu})(x,\xi,x')$ of $Q_{\epsilon\nu}$ is included in $B_\epsilon = \{(x,\xi,x') |\ \epsilon(x,\xi,x') \in B\}$. Thus for this $Q_{\epsilon\nu}$, (5.92) also holds:

$$\|Q_{\epsilon\nu} f\| \le V_\epsilon^2 |\sigma(Q_{\epsilon\nu})|_0 \|f\| \quad (f \in \mathcal{S}). \tag{5.93}$$

By $\|Q_{\epsilon\nu} f\|^2 = (Q_{\epsilon\nu}^* Q_{\epsilon\nu} f, f) \le \|Q_{\epsilon\nu}^* Q_{\epsilon\nu}\| \|f\|^2$, we have $\|Q_{\epsilon\nu}\|^2 \le \|Q_{\epsilon\nu}^* Q_{\epsilon\nu}\| = \|Q_{\epsilon(2\nu)}\|$. Thus repeating this argument, we have

$$\|P_\epsilon\|^2 \le \|P_\epsilon^* P_\epsilon\| = \|Q_{\epsilon 1}\| \le \|Q_{\epsilon 2}\|^{1/2} \le \cdots \le \|Q_{\epsilon(2^\ell)}\|^{1/2^\ell}$$

for any $\ell = 0, 1, 2, \cdots$. Inserting (5.93) into this and applying (5.90), we obtain

$$\|P_\epsilon\| \le (V_\epsilon^2 |\sigma(Q_{\epsilon 2^\ell})|_0)^{1/2^{\ell+1}} \le V_\epsilon^{1/2^\ell} C_0 |p_\epsilon|_{3m_0}.$$

Letting $\ell \to \infty$ we get

$$\|P_\epsilon\| \le C_0 |p_\epsilon|_{3m_0}.$$

Since, as $\epsilon \to 0$, $P_\epsilon f \to Pf$ in $\mathcal{H}$ for $f \in \mathcal{S}$ and $|p_\epsilon|_{3m_0} \to |p|_{3m_0}$, we obtain the theorem. □

With these preparations, we investigate the so-called propagation estimates for e^{-itH_0}. Propagation estimates are the estimates with respect to time t of the operator norm of

$$P_1 e^{-itH_0} P_2,$$

where P_j are the ψdo's of the following types:

$$P_s = \langle x \rangle^{-s}, \quad P_s q(D_x) \quad (s \ge 0) \tag{5.94}$$

$$P_+ f(x) = \text{Os-}\int\int e^{i(x-y)\xi} p_+(\xi, y) f(y) dy d\widehat{\xi}, \tag{5.95}$$

$$P_- f(x) = \text{Os-}\int\int e^{i(x-y)\xi} p_-(x, \xi) f(y) dy d\widehat{\xi}. \tag{5.96}$$

Here q and $p_\pm$ are the symbols satisfying for all $\ell = 0, 1, 2, \cdots$

$$|q|_\ell + |p_\pm|_\ell < \infty$$

and for some $\theta \in (-1,1)$ and $\rho > 0$ with $\theta + \rho < 1$ and $\theta - \rho > -1$

$$|\partial_x^\alpha \partial_\xi^\beta p_+(\xi,x)| \le C_{\ell\alpha\beta}\langle x\rangle^{-\ell} \quad \left(\cos(x,\xi) := \frac{x\xi}{|x||\xi|} < \theta + \rho\right) \tag{5.97}$$
$$|\partial_x^\alpha \partial_\xi^\beta p_-(x,\xi)| \le C_{\ell\alpha\beta}\langle x\rangle^{-\ell} \quad (\cos(x,\xi) > \theta - \rho). \tag{5.98}$$

For a technical reason to avoid the singularities at $x = 0$ and $\xi = 0$, we also assume for some $\sigma > 0$

$$q(\xi) = 0 \quad (|\xi| < \sigma) \tag{5.99}$$
$$p_+(\xi,x) = 0, \quad p_-(x,\xi) = 0 \quad (|x| < \sigma \text{ or } |\xi| < \sigma). \tag{5.100}$$

Of course by Proposition 5.3, we can consider other forms of ψdo's by rewriting them as the ψdo's of the form of (5.95) or (5.96).

What we prove are the followings:

Theorem 5.5 *Let P_s be as above. Then we have for any $s \ge 0$*

$$\|P_s q(D_x) e^{-itH_0} P_s\| \le C_s \langle t\rangle^{-s} \quad (t \in R^1), \tag{5.101}$$

where the constant $C_s > 0$ is independent of $t \in R^1$.

Theorem 5.6 *Let P_s and $P_\pm$ be as above. Then we have for any $s \ge 0$ and $s \ge \delta \ge 0$*

$$\|P_s e^{-itH_0} P_+ \langle x\rangle^\delta\| \le C_{s\delta}\langle t\rangle^{-s+\delta} \quad (t \ge 0), \tag{5.102}$$
$$\|\langle x\rangle^\delta P_- e^{-itH_0} P_s\| \le C_{s\delta}\langle t\rangle^{-s+\delta} \quad (t \ge 0), \tag{5.103}$$

where the constant $C_{s\delta} > 0$ is independent of t.

Theorem 5.7 *Let $P_\pm$ be as above. Then we have for any $s \ge 0$ and $\delta \ge 0$*

$$\|\langle x\rangle^\delta P_- e^{-itH_0} P_+ \langle x\rangle^\delta\| \le C_{s\delta}\langle t\rangle^{-s} \quad (t \ge 0), \tag{5.104}$$
$$\|\langle x\rangle^\delta P_+ e^{-itH_0} P_- \langle x\rangle^\delta\| \le C_{s\delta}\langle t\rangle^{-s} \quad (t \le 0), \tag{5.105}$$

where the constant $C_{s\delta} > 0$ is independent of t.

If we take a Laplace transform of (5.101) when $s > 1$, we obtain for $\epsilon > 0$

$$\left\| i\int_0^\infty P_s q(D_x) e^{it(\lambda+i\epsilon)} e^{-itH_0} P_s dt \right\| \le C_s,$$

where $C_s > 0$ is independent of $\epsilon > 0$. Since the integral is equal to $P_s q(D_x) R_0(\lambda+i\epsilon) P_s = \langle x\rangle^{-s} q(D_x)(H_0 - (\lambda + i\epsilon))^{-1}\langle x\rangle^{-s}$, this implies a partial result of Theorem 5.1 for $s > 1$.

Also taking the Laplace transform of (5.104) and (5.105), we obtain for $\delta \ge 0$

$$\|P_\mp R_0(\lambda \pm i\epsilon) P_\pm\|_{L^2_{-\delta} \to L^2_\delta} \le C_\delta, \tag{5.106}$$

where $C_\delta > 0$ is independent of $\epsilon > 0$.

The reason that these theorems are called propagation estimates is as follows: In the case of Theorem 5.5, P_s on the RHS of $P_s e^{-itH_0} P_s f$ when applied to a function $f \in \mathcal{H} = L^2(R^m)$ restricts the initial state f in a localized region around 0 to the order $\langle x\rangle^{-s}$. P_s on the LHS restricts the propagated state $e^{-itH_0}P_s f$ also in a localized region around 0 to the same extent. According to the propagator e^{-itH_0}, as is indicated by Theorem 3.2, the wave function f restricted to a region G in R^m_x should propagate along a line parallel to the "idealized" velocity v with $|v| \geq \sigma(>0)$: If f is localized in a region G of R^m, physically the region G should move to the region $G+tv = \{x+tv|\ x \in G\}$ after time t. Thus the localization factor P_s on the LHS should effect so that $\|P_s e^{-itH_0} P_s f\|$ decays as $t \to \infty$. Theorem 5.5 tells that the rate of this decay is t^{-s}.

In the case of (5.102) of Theorem 5.6, P_+ on the RHS of $P_s e^{-itH_0} P_+$ restricts the initial function to the phase space region where $\cos(x,v) \geq \theta + \rho$ by (5.97). Then the state $e^{-itH_0}P_+ f$ propagates toward the direction $v \neq 0$ which is almost parallel to x. Thus the location of the state should be separated from 0 when $t \to \infty$. Theorem 5.6 gives the rate of this separation.

In the case of Theorem 5.7, similarly to (5.102) of Theorem 5.6, the state $e^{-itH_0}P_+ f$ propagates toward the region in the phase space where x and v are almost parallel to each other. Since P_- on the LHS restricts the state to the region where x and v are anti-parallel, the state $P_- e^{-itH_0}P_+ f$ should decay. Theorem 5.7 gives the rate of this decay.

We now prove these theorems.

Proof of Theorem 5.5: It suffices to prove (5.101) for even integers $s \geq 0$, because we have only to interpolate them to get the desired estimates. The case $s = 0$ is obvious. Let $s > 0$ be an even integer. We recall (5.49):

$$e^{-itH_0}f(x) = \text{Os-}\int\int e^{i(x\xi - t|\xi|^2/2 - \xi y)} f(y)dy\widehat{d\xi}, \tag{5.107}$$

and the arguments after it, where it was shown that (5.107) is rewritten as (5.53) in which integration by parts with respect to ξ and y can be performed any times. Thus using the relation

$$(1 - t\xi D_\xi)e^{-it\xi^2/2} = (1 + |t\xi|^2)e^{-it\xi^2/2} =: h(t,\xi)e^{-it\xi^2/2},$$

we integrate by parts in (5.107) to get

$$\begin{aligned} q(D_x)e^{-itH_0}f(x) &= \text{Os-}\int\int e^{-it|\xi|^2/2}\left((1 + t\xi D_\xi)h(t,\xi)^{-1}\right)^s [q(\xi)e^{-i\xi y}e^{ix\xi}]f(y)dy\widehat{d\xi} \\ &= \text{Os-}\int\int e^{i(x\xi - t|\xi|^2/2)}\sum_{j=1}^{J} Q_j(x)h_j(t,\xi)P_j(y)e^{-i\xi y}f(y)dy\widehat{d\xi}. \end{aligned}$$

Here J is an integer, $Q_j(x)$ and $P_j(y)$ are polynomials of x and y of order up to s, and $h_j(t,\xi)$ satisfies

$$|h_j(t,\xi)|_\ell \leq C_\ell \langle t\rangle^{-s} \quad (\ell = 0, 1, 2, \cdots)$$

by the inequality

$$(1+|t\xi|^2)^{-s/2} \le C\langle t\rangle^{-s},$$

which holds on supp q by (5.99). From these, we obtain

$$P_s q(D_x)e^{-itH_0}P_s f(x) = \sum_{j=1}^{J} \text{Os-}\int\int e^{i(x\xi - t|\xi|^2/2 - \xi y)} u_j(x)h_j(t,\xi)s_j(y)f(y)dy d\widehat{\xi}, \quad (5.108)$$

where $u_j(x)$ and $s_j(y)$ satisfy

$$|u_j|_\ell + |s_j|_\ell < \infty$$

for all $\ell = 0, 1, 2, \cdots$. Noting that (5.108) can be written as

$$P_s q(D_x)e^{-itH_0}P_s f = \sum_{j=1}^{J} u_j(X)h_j(t, D_x)e^{-itH_0}s_j(X')f,$$

we use Theorem 5.4 to conclude the proof. □

Proof of Theorem 5.6: We prove (5.102) only. Another is proved similarly. We have only to prove the estimate for an even integer $s \ge 0$ as above. We write

$$e^{-itH_0}P_+\langle x\rangle^\delta f(x) = \text{Os-}\int\int e^{ix\xi}e^{-i(t\xi^2/2+\xi y)}p_+(\xi, y)\langle y\rangle^\delta f(y)dy d\widehat{\xi},$$

and integrate by parts with respect to ξ using the relation

$$(1+|t\xi+y|^2)^{-1}(1-(t\xi+y)D_\xi)e^{-i(t\xi^2/2+\xi y)} = e^{-i(t\xi^2/2+\xi y)}.$$

Noting

$$\begin{aligned}&(1+|t\xi+y|)^{-1}(1+|y|)^{-1} \le C(1+|t\xi|)^{-1},\\ &|t\xi+y|^{-1} \le C(|t\xi|+|y|)^{-1} \quad (\cos(y,\xi) \ge \theta+\rho)\end{aligned}$$

and applying (5.97) and (5.100) to the result of the integration by parts, we obtain

$$e^{-itH_0}P_+\langle x\rangle^\delta f(x) = \sum_{k=1}^{K} P_k(x)e^{-itH_0}s_k(t; D_x, X')\langle y\rangle^\delta f(y), \quad (5.109)$$

where K is some integer, $P_k(x)$ is a polynomial of x of order at most s, and each symbol $s_k(t;\xi,y)$ satisfies

$$|s_k(t;\xi,y)|_\ell \le C_\ell\langle t\rangle^{-s+\delta}$$

for all $\ell = 0, 1, 2, \cdots$. Thus P_s on the LHS of (5.102) damps the growth of $P_k(x)$, and we obtain the desired estimate by using Theorem 5.4. □

Proof of Theorem 5.7: We consider the case $t \geq 0$ only. The other case is proved similarly. We divide P_- and P_+ as

$$P_- = P_{--} + P_{-+}, \quad P_+ = P_{+-} + P_{++},$$

where the respective symbols satisfy

$$\begin{aligned}
&\text{supp } p_{--}(x,\xi) \subset \{(x,\xi)|\ \cos(x,\xi) < \theta - \rho/3, |x| \geq \sigma, |\xi| \geq \sigma\} \\
&\text{supp } p_{-+}(x,\xi) \subset \{(x,\xi)|\ \cos(x,\xi) > \theta - 2\rho/3, |x| \geq \sigma, |\xi| \geq \sigma\} \\
&\text{supp } p_{+-}(\xi,y) \subset \{(\xi,y)|\ \cos(y,\xi) < \theta + 2\rho/3, |y| \geq \sigma, |\xi| \geq \sigma\} \\
&\text{supp } p_{++}(\xi,y) \subset \{(\xi,y)|\ \cos(y,\xi) > \theta + \rho/3, |y| \geq \sigma, |\xi| \geq \sigma\}
\end{aligned}$$

and

$$|\langle x\rangle^k p_{-+}|_\ell < \infty, \quad |\langle y\rangle^k p_{+-}|_\ell < \infty \tag{5.110}$$

for any $k, \ell = 0, 1, 2, \cdots$. Then we have to estimate the four terms:

$$\begin{aligned}
&\langle x\rangle^\delta P_{--} e^{-itH_0} P_{+-} \langle x\rangle^\delta, \quad \langle x\rangle^\delta P_{--} e^{-itH_0} P_{++} \langle x\rangle^\delta, \\
&\langle x\rangle^\delta P_{-+} e^{-itH_0} P_{+-} \langle x\rangle^\delta, \quad \langle x\rangle^\delta P_{-+} e^{-itH_0} P_{++} \langle x\rangle^\delta.
\end{aligned}$$

By Theorem 5.6 and (5.110), we have

$$\begin{aligned}
&\|\langle x\rangle^\delta P_{-+} e^{-itH_0} P_{+-} \langle x\rangle^\delta\| \leq C_{s\delta} \langle t\rangle^{-s}, \\
&\|\langle x\rangle^\delta P_{-+} e^{-itH_0} P_{++} \langle x\rangle^\delta\| \leq C_{s\delta} \langle t\rangle^{-s}, \\
&\|\langle x\rangle^\delta P_{--} e^{-itH_0} P_{+-} \langle x\rangle^\delta\| \leq C_{s\delta} \langle t\rangle^{-s}.
\end{aligned}$$

Thus we have only to consider

$$\begin{aligned}
&\langle x\rangle^\delta P_{--} e^{-itH_0} P_{++} \langle x\rangle^\delta f(x) \\
&= \text{Os-}\int\int e^{i(x\xi - t\xi^2/2 - y\xi)} \langle x\rangle^\delta p_{--}(x,\xi) p_{++}(\xi,y) \langle y\rangle^\delta f(y) dy d\hat{\xi}.
\end{aligned} \tag{5.111}$$

Noting

$$|x - t\xi - y|^{-1} \leq C(|x| + |t\xi| + |y|)^{-1} \tag{5.112}$$

for (x,ξ,y) satisfying $\cos(x,\xi) \leq \theta - \rho/3$ and $\cos(\xi,y) \geq \theta + \rho/3$, we integrate by parts in (5.111) by using the relation

$$Qe^{i(x\xi - t\xi^2/2 - y\xi)} = e^{i(x\xi - t\xi^2/2 - y\xi)},$$

where

$$Q = (1 + |x - t\xi - y|^2)^{-1}(1 + (x - t\xi - y)D_\xi).$$

Then we obtain

$$\langle x\rangle^\delta P_{--} e^{-itH_0} P_{++} \langle x\rangle^\delta f(x) = \text{Os-}\int\int e^{i(x-y)\xi} r_t^{(k)}(x,\xi,y) f(y) dy d\hat{\xi},$$

where

$$r_t^{(k)}(x,\xi,y) = e^{-it\xi^2/2}({}^tQ)^k(\langle x\rangle^\delta p_{--}(x,\xi)p_{++}(\xi,y)\langle y\rangle^\delta)$$

satisfies by (5.112)

$$|\partial_x^\alpha\partial_\xi^\beta\partial_y^\gamma r_t^{(k)}(x,\xi,y)| \le C_{\alpha\beta\gamma k}\langle t\xi\rangle^{3m_0}\langle x\rangle^{-k/3+\delta}\langle t\xi\rangle^{-k/3}\langle y\rangle^{-k/3+\delta}$$

for all α,β,γ with $|\alpha+\beta+\gamma| \le 3m_0$ and any integer $k \ge 0$. Therefore by Theorem 5.4 and (5.100), we obtain the theorem. □

Chapter 6

Two-Body Hamiltonian

6.1 Eigenvalues of a two-body Hamiltonian

In this chapter we consider a perturbed Hamiltonian defined in $\mathcal{H} = L^2(R^m)$ ($m = 1, 2, \cdots$)

$$H = H_0 + V, \tag{6.1}$$

where H_0 is the free Hamiltonian defined by (5.1) and V is a multiplication operator by a real-valued measurable function $V(x)$ that can be decomposed as a sum of two real-valued measurable functions: $V(x) = V_S(x) + V_L(x)$ satisfying the decay assumption:

$$|V_S(x)| \le C\langle x\rangle^{-1-\delta}, \tag{6.2}$$
$$|\partial_x^\alpha V_L(x)| \le C_\alpha \langle x\rangle^{-|\alpha|-\delta} \tag{6.3}$$

for a constant $0 < \delta < 1$ and all multi-indices α with constants $C > 0$ and $C_\alpha > 0$ independent of $x \in R^m$. H is a generalization of a two-body Hamiltonian that describes the two-body system in R^3 to a general dimension $m = 1, 2, \cdots$. V_S and V_L are called short- and long-range potentials respectively. The assumption (6.2) can be weakened to allow some local singularities with respect to $x \in R^m$. E.g.

$$h(R) = \|V_S(H_0 + 1)^{-1}\chi_{\{x||x|>R\}}\| \in L^1((0, \infty)) \tag{6.4}$$

is known sufficient ([8]) for some results we prove below. (6.4) includes the Coulomb singularities of order $1/|x|$ at $x = 0$, thus together with the long-range part V_L, V covers the Coulomb type long-range potentials. As the inclusion of singularities is of rather technical nature, we restrict the description below to the potentials satisfying (6.2) and (6.3).

As is well-known (see, e.g. [43], Chapter 4.), the perturbed Hamiltonian (6.1) has eigenvalues in general, unlike the unperturbed Hamiltonian H_0. Thus we have first to specify eigenvalues and eigenspace $\mathcal{H}_p(H)$ of H. Then restricting the total Hilbert space $\mathcal{H}$ to the continuous spectral subspace $\mathcal{H}_c(H) = \mathcal{H}_p(H)^\perp$, we will consider the properties of scattering states $f \in \mathcal{H}_c(H)$.

Before going to the investigation of eigenvalues of H, we see that H defines a selfadjoint operator in $\mathcal{H}$. In the case of the unperturbed Hamiltonian H_0, the selfadjointness was trivial by virtue of the expression (5.5). But in the case of the perturbed Hamiltonian, getting such an expression is a problem that will be solved later. We recall the definition of the adjoint operator H^* of H. Let $\mathcal{D}(H^*)$ be defined as the set of all $g \in \mathcal{H}$ such that there exists an $f \in \mathcal{H}$ satisfying

$$(g, Hu) = (f, u) \quad \text{for all } u \in \mathcal{D}(H). \tag{6.5}$$

Since $\mathcal{D}(V) = \mathcal{H}$, the domain $\mathcal{D}(H)$ of the Hamiltonian $H = H_0 + V$ is equal to $\mathcal{D}(H_0) = H^2(R^m)$, which is dense in $\mathcal{H}$. Thus the relation (6.5) determines f uniquely. Then we define $H^*g = f$ for $g \in \mathcal{D}(H^*)$. In general, H is called symmetric if H^* is an extension of H (in notation $H \subset H^*$) and selfadjoint when $H^* = H$. A symmetric operator H is called essentially selfadjoint if the closure H^{**} of H is selfadjoint. In our case of H_0 and V, it is clear that H_0 and V are selfadjoint with $\mathcal{D}(H_0^*) = H^2(R^m) = \mathcal{D}(H_0)$ by (5.5), and $\mathcal{D}(V^*) = \mathcal{H} = \mathcal{D}(V)$ by our assumptions (6.2) and (6.3). Thus $\mathcal{D}(H^*) = \mathcal{D}(H_0^*) \cap \mathcal{D}(V^*) = H^2(R^m) = \mathcal{D}(H)$; hence H is selfadjoint.

Turning to the eigenvalues of H, we define $i[H, A]$ as a form sum

$$(i[H, A]f, g) = i(Af, Hg) - i(Hf, Ag) \tag{6.6}$$

for $f, g \in \mathcal{S}$, where as before $A = (x \cdot D_x + D_x \cdot x)/2 = x \cdot D_x + m/(2i) = D_x \cdot x - m/(2i)$ is a selfadjoint operator with $\mathcal{D}(A) = H_1^1(R^m)$, the weighted Sobolev space, as defined in the section of Notation at the beginning.

We first prove the non-existence of positive eigenvalues of H. To do so we assume that $u \in \mathcal{D}(H)$ satisfies $Hu = \lambda u$ for some $\lambda > 0$, and will prove that $u = 0$. Then we have shown that H has no positive eigenvalues. We set $P(\lambda) = E_H(\lambda) - E_H(\lambda - 0)$ for the above $\lambda > 0$. (See the section of Notation.) Then letting $B = (\lambda - \epsilon/\mu, \lambda + \epsilon/\mu)$ with $\epsilon > 0$ sufficiently small and $\mu > 1$ sufficiently large, and using the equalities as form sums $i[H_0, A] = 2H_0$ and $i[V_L, A] = -x \cdot \nabla_x V_L(x)$, we have

$$\begin{aligned}
&(E_H(B)i[H, A]E_H(B)u, u) \\
&= i(AE_H(B)u, HE_H(B)u) - i(HE_H(B)u, AE_H(B)u) \\
&= i(AE_H(B)u, V_S E_H(B)u) - i(V_S E_H(B)u, AE_H(B)u) \\
&\qquad + (i[V_L, A]E_H(B)u, E_H(B)u) + (2H_0 E_H(B)u, E_H(B)u) \\
&= i(x \cdot D_x E_H(B)u, V_S E_H(B)u) - i(V_S E_H(B)u, x \cdot D_x E_H(B)u) \\
&\qquad + m(V_S E_H(B)u, E_H(B)u) - (x \cdot \nabla_x V_L E_H(B)u, E_H(B)u) \\
&\qquad - (2V E_H(B)u, E_H(B)u) + (2H E_H(B)u, E_H(B)u).
\end{aligned} \tag{6.7}$$

By (6.2) and (6.3), $E_H(B)V_S(x \cdot D_x)E_H(B)$, $E_H(B)V_S E_H(B)$, $E_H(B)(x \cdot \nabla_x V_L)E_H(B)$ and $E_H(B)VE_H(B)$ are compact operators defined on $\mathcal{H}$. We now assume that $\varphi(x/R)u(x)$ is not identically 0 for any $R > 0$, where $\varphi \in C^\infty(R^m)$, $0 \le \varphi \le 1$, $\varphi(x) = 1$ for $|x| \ge 2$ and $\varphi(x) = 0$ for $|x| \le 1$. We denote by φ_R the multiplication operator by $\varphi(x/R)$. In (6.7), we replace u by $u_R = \varphi_R u / \|\varphi_R u\|$ with $R > 1$. Then since $u_R \to 0$ weakly in $\mathcal{H}$ as $R \to \infty$, by taking a large $R > 1$ we can bound the terms on the RHS of (6.7) except for the last term by an arbitrarily small constant times $\|u_R\|^2$, while the last term is bounded by $2(\lambda - \epsilon/\mu)(E_H(B)u_R, E_H(B)u_R) = 2(\lambda - \epsilon/\mu)\|E_H(B)u_R\|^2$ from below. Thus we have

$$\liminf_{R\to\infty} (E_H(B)i[H, A]E_H(B)u_R, u_R) \ge \alpha \liminf_{R\to\infty} \|E_H(B)u_R\|^2 \tag{6.8}$$

for some constant $\alpha > 0$. We remark that this inequality holds uniformly in small $\epsilon > 0$ and large $\mu > 1$.

On the other hand, if we set $R_\mu = \mu(\mu + iA)^{-1}$ for the same $\mu > 1$, it is easy to see that $\sup_{\mu>1} \|R_\mu\| \le 1$, and $\text{s-lim}_{\mu\to\infty} R_\mu = I$. Then we have that for any small $\epsilon > 0$ there is an $M_\epsilon > 1$ such that for $\mu > M_\epsilon$:

$$\begin{aligned}
&|(E_H(B)i[H, R_\mu]E_H(B)u_R, u_R)| \qquad (6.9)\\
&= |i(R_\mu E_H(B)u_R, (H-\lambda)E_H(B)u_R) - i((H-\lambda)E_H(B)u_R, R_\mu E_H(B)u_R)|\\
&\le 2\epsilon/\mu \|E_H(B)u_R\|^2.
\end{aligned}$$

By a calculation, we have $-\mu[H, R_\mu] = R_\mu i[H, A]R_\mu$. Thus we have for $\epsilon > 0$ and $\mu > M_\epsilon$

$$|(E_H(B)R_\mu i[H, A]R_\mu E_H(B)u_R, u_R)| \le 2\epsilon\|E_H(B)u_R\|^2. \qquad (6.10)$$

Letting $\mu \to \infty$, we obtain for an arbitrarily small $\epsilon > 0$

$$|(P(\lambda)i[H, A]P(\lambda)u_R, u_R)| \le 2\epsilon\|P(\lambda)u_R\|^2. \qquad (6.11)$$

This and (6.8) with being let $\mu \to \infty$ yield a contradiction. Therefore our assumption that $\varphi(x/R)u(x)$ is not identically 0 for any $R > 0$ is false, and we have proved that there is an $R > 0$ such that $\varphi(x/R)u(x) = 0$ for all $x \in R^m$. In particular we have that $u(x) = 0$ for $|x| \ge 2R$. Recalling that u satisfies $Hu = \lambda u$ and the well-known unique continuation theorem for the solutions of elliptic partial differential equations (see e.g., [14]), we see that $u(x) = 0$ for all $x \in R^m$. Hence H has no positive eigenvalues.

Next we consider negative eigenvalues of H. Assume that $Hu_j = \lambda_j u_j$ with $\lambda_j \le \lambda_0 < 0$ and $(u_i, u_j)_{\mathcal{H}} = \delta_{ij}$ for $i, j = 1, 2, \cdots$ and that $\lambda_j \to \lambda(\le \lambda_0)$ as $j \to \infty$. Then

$$(H_0 - \lambda_j)u_j = -Vu_j \in L^2_\delta$$

by our assumptions (6.2) and (6.3). Since $-\lambda_j \ge -\lambda_0 > 0$, from this relation we obtain

$$\sup_j \|u_j\|_{H^2_\delta} = \sup_j \|(H_0 - \lambda_j)^{-1}Vu_j\|_{H^2_\delta} < \infty. \qquad (6.12)$$

In particular, we have that $\{u_j\}_{j=1}^\infty$ forms a precompact subset of $\mathcal{H} = L^2(R^m)$. Therefore there is a subsequence $\{u_{j_k}\}$ of $\{u_j\}$ such that $u_{j_k} \to u$ as $k \to \infty$ in $\mathcal{H}$ for some $u \in \mathcal{H}$. By $\|u_{j_k}\|_{\mathcal{H}} = 1$, we thus have $\|u\|_{\mathcal{H}} = 1$ and $\lim_{k\to\infty}(u_{j_k}, u)_{\mathcal{H}} = \|u\|^2_{\mathcal{H}} = 1$. On the other hand by our assumption $(u_i, u_j)_{\mathcal{H}} = \delta_{ij}$, we also have $(u_{j_k}, u)_{\mathcal{H}} = \lim_{\ell\to\infty}(u_{j_k}, u_{j_\ell}) = 0$ for any $k = 1, 2, \cdots$, a contradiction. Thus no negative eigenvalues accumulate to a real number other than 0. Further taking $\lambda_j = \lambda \le \lambda_0 < 0$ in the argument, we also see that all negative eigenvalues have finite multiplicity.

Summarizing we have proved:

Theorem 6.1 *Let the conditions (6.2) and (6.3) be satisfied. Then the two-body Hamiltonian $H = H_0 + V$ in (6.1) has no positive eigenvalues, and its negative eigenvalues are of finite multiplicity and do not accumulate other than* 0. *In particular, the set of eigenvalues of H is at most countable, and discrete except for the neighborhoods of* 0.

6.2 Wave operators

We now consider the behavior of $\exp(-itH)f$ for $f \in \mathcal{H}_c(H) = \mathcal{H}_p(H)^\perp$ when time t varies. As has been indicated in Theorem 3.2, $\exp(-itH)f$ for all $f \in \mathcal{H}_c(H)$ scatter as $t \to \pm\infty$. In the case of the two-body Hamiltonian H, the theorem implies for any $f \in \mathcal{H}_c(H) \cap H^2(R^m) \cap L^2_2(R^m)$, $R > 0$, and $\phi \in C_0^\infty(R^1)$

$$\|F(|x| < R)\exp(-it_m H)f\| \to 0, \tag{6.13}$$

$$\|(\phi(H) - \phi(H_0))\exp(-it_m H)f\| \to 0, \tag{6.14}$$

$$\left\|\left(\frac{x}{t_m} - D_x\right)\exp(-it_m H)f\right\| \to 0 \tag{6.15}$$

as $m \to \pm\infty$. We note that (6.14) implies $E_H(B)f = 0$ for $f \in \mathcal{H}_c(H)$ and $B \subset (-\infty, 0)$, which is seen by taking $\phi \in C_0^\infty((-\infty,0))$ in (6.14) and remembering $H_0 \geq 0$. (6.15) yields that the relative position of the two particles goes to the direction almost parallel to the relative momentum of the two particles. This is an analogue of the propagation estimates Theorems 5.5-5.7 for the free Hamiltonian. Thus we are tempted to compare the behavior of $\exp(-itH)f$ ($f \in \mathcal{H}_c(H)$) with the behavior of the free propagator $\exp(-itH_0)$ as $t \to \pm\infty$. In fact we can prove the following theorem:

Theorem 6.2 *Let (6.2) be satisfied and let $V_L(x) = 0$ ($x \in R^m$). Then there exist the limits*

$$W_\pm g = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}g \tag{6.16}$$

for any $g \in \mathcal{H} = L^2(R^m)$. $W_\pm$ are called the wave operators.

Proof: We compute

$$e^{itH}e^{-itH_0}g - g = \int_0^t \frac{d}{ds}(e^{isH}e^{-isH_0}g)ds = i\int_0^t e^{isH}V_S e^{-isH_0}g ds. \tag{6.17}$$

Since the functions of the form $g = q(D_x)\langle x\rangle^{-1-\delta}h$ with $h \in L^2(R^m)$ and q being the symbol satisfying (5.99) in section 5.3 are dense in $\mathcal{H} = L^2(R^m)$ and the operator $e^{itH}e^{-itH_0}$ is uniformly bounded in $t \in R^1$, we have only to show the convergence of the integral in (6.17) for $g = q(D_x)\langle x\rangle^{-1-\delta}h$, where $\delta > 0$ is the constant in (6.2). By Theorem 5.5 and (6.2), we have

$$\|V_S e^{-isH_0}q(D_x)\langle x\rangle^{-1-\delta}h\| \leq \langle s\rangle^{-1-\delta}\|h\| \in L^1((-\infty,\infty)). \tag{6.18}$$

Thus the integral in (6.17) converges for such a g and the proof is complete. □

By the definition (6.16) of $W_\pm$, we easily see that $W_\pm$ are isometric operators from $\mathcal{H}$ into $\mathcal{H}$ and the relation holds:

$$e^{isH}W_\pm = W_\pm e^{isH_0} \quad (s \in R^1). \tag{6.19}$$

By (5.56) and its variant for H, we have for $\epsilon > 0, \lambda \in R^1$ and $f \in \mathcal{H}$

$$R_0(\lambda \pm i\epsilon)f = i\int_0^{\pm\infty} e^{is(\lambda \pm i\epsilon - H_0)} f ds \tag{6.20}$$

$$R(\lambda \pm i\epsilon)f = i\int_0^{\pm\infty} e^{is(\lambda \pm i\epsilon - H)} f ds. \tag{6.21}$$

Taking the Laplace transforms of both sides of (6.19), we thus have

$$R(\lambda \pm i\epsilon)W_\pm = W_\pm R_0(\lambda \pm i\epsilon). \tag{6.22}$$

It follows from this that

$$\frac{1}{2\pi i}(R(\lambda + i\epsilon) - R(\lambda - i\epsilon))W_\pm f = W_\pm \frac{1}{2\pi i}(R_0(\lambda + i\epsilon) - R_0(\lambda - i\epsilon)). \tag{6.23}$$

Let $\overline{E}_H(a) = \frac{1}{2}(E_H(a-0) + E_H(a))$, etc. Then it is not difficult to see (see [50] p. 325 or [21] p.359) by using (5.10)-(5.12) that for $-\infty < a < b < \infty$

$$\text{s-}\lim_{\epsilon \downarrow 0} \frac{1}{2\pi i}\int_a^b (R(\lambda + i\epsilon) - R(\lambda - i\epsilon))d\lambda = \overline{E}_H(b) - \overline{E}_H(a), \tag{6.24}$$

$$\text{s-}\lim_{\epsilon \downarrow 0} \frac{1}{2\pi i}\int_a^b (R_0(\lambda + i\epsilon) - R_0(\lambda - i\epsilon))d\lambda = \overline{E}_0(b) - \overline{E}_0(a). \tag{6.25}$$

From these and (6.23), we now have

$$E_H(B)W_\pm = W_\pm E_0(B) \tag{6.26}$$

for any Borel set $B \subset (-\infty, \infty)$, where $E_H(B)$ and $E_0(B)$ are spectral measures for H and H_0 respectively. From (6.26), we have for any continuous function $F(\lambda)$,

$$F(H)W_\pm \supset W_\pm F(H_0). \tag{6.27}$$

This relation is called the intertwining property of the wave operators $W_\pm$.

We define the absolutely continuous spectral subspace $\mathcal{H}_{ac}(H)$ of a selfadjoint operator H as the space of functions $f \in \mathcal{H}$ such that

$$E_H(B)f = 0 \quad \text{if} \quad |B| = 0, \tag{6.28}$$

where $|B|$ is the Lebesgue measure of a Borel set B of R^1. We remark that $f \in \mathcal{H}_c(H) = \mathcal{H}_p(H)^\perp$ is equivalent to the condition

$$(E_H(\lambda)f, f) \text{ is continuous with respect to } \lambda \in R^1. \tag{6.29}$$

In fact, the eigenspace of H for an eigenvalue $\lambda \in R^1$ is equal to $P(\lambda)\mathcal{H} := (E_H(\lambda) - E_H(\lambda - 0))\mathcal{H}$, where $E_H(\lambda - 0) = \text{s-}\lim_{\mu\uparrow\lambda} E_H(\mu)$. Thus $\mathcal{H}_p(H)$ is spanned by $P(\lambda)\mathcal{H}$ with $\lambda \in R^1$, and the orthogonal complement $\mathcal{H}_p(H)^\perp$ consists of those elements f that satisfy (6.29). In the case of the free Hamiltonian H_0, it follows from Theorem 5.1 and $\mathcal{H}_p(H_0) = \{0\}$ that

$$\mathcal{H}_{ac}(H_0) = \mathcal{H}_c(H_0) = \mathcal{H}. \tag{6.30}$$

Combining this with the intertwining property (6.26), we see that for $f = W_\pm g$ in the range $\mathcal{R}(W_\pm) = W_\pm \mathcal{H}$,

$$E_H(B)f = E_H(B)W_\pm g = W_\pm E_0(B)g = 0 \tag{6.31}$$

for a Borel set B with the Lebesgue measure $|B| = 0$. Namely $W_\pm f$ $(f \in \mathcal{H})$ belongs to the absolutely continuous spectral subspace $\mathcal{H}_{ac}(H)(\subset \mathcal{H}_c(H))$ of H. Thus we have shown

$$\mathcal{R}(W_\pm) \subset \mathcal{H}_{ac}(H) \subset \mathcal{H}_c(H). \tag{6.32}$$

For the perturbed Hamiltonian H, $\mathcal{H}_p(H) \neq \{0\}$ in general as seen above, thus (6.30) does not necessarily hold. Nevertheless by Theorem 3.2, it is expected that

$$\mathcal{R}(W_\pm) = \mathcal{H}_{ac}(H) = \mathcal{H}_c(H) \tag{6.33}$$

holds. If this equality holds, $\mathcal{H}_c(H)$ is unitarily equivalent to $\mathcal{H}_c(H_0) = \mathcal{H} = L^2(R^m)$ by the unitary operators

$$W_\pm : \mathcal{H} = \mathcal{H}_c(H_0) = \mathcal{H}_{ac}(H_0) \longrightarrow \mathcal{H}_c(H) = \mathcal{H}_{ac}(H), \tag{6.34}$$

which intertwine H and H_0. The relation (6.33) is called the *asymptotic completeness* of the wave operators $W_\pm$. Assuming the asymptotic completeness holds, we define unitary operators

$$\mathcal{F}_\pm = \mathcal{F}W_\pm^* : \mathcal{H}_c(H) \longrightarrow \mathcal{F}\mathcal{H}, \tag{6.35}$$

where $\mathcal{F}$ is Fourier transformation as before. Then by the intertwining property (6.27), we have for $f \in \mathcal{H}_c(H) \cap \mathcal{D}(H)$

$$\mathcal{F}_\pm Hf(\xi) = \mathcal{F}W_\pm^* Hf(\xi) = \mathcal{F}H_0 W_\pm^* f(\xi) = \frac{|\xi|^2}{2}\mathcal{F}_\pm f(\xi). \tag{6.36}$$

Thus $\mathcal{F}_\pm$ are considered as generalizations of Fourier transformation that diagonalize H on $\mathcal{H}_c(H)$ as the usual Fourier transformation $\mathcal{F}$ diagonalizes H_0: $\mathcal{F}H_0 f(\xi) = (|\xi|^2/2)\mathcal{F}f(\xi)$ $(f \in \mathcal{H}_c(H_0) \cap \mathcal{D}(H_0) = \mathcal{D}(H_0))$. Thus the asymptotic completeness gives a spectral representation of the perturbed Hamiltonian $H = H_0 + V$.

6.3 Asymptotic completeness

In this section, we will be devoted in the proof of the asymptotic completeness (6.33). To do so, since by definition

$$\mathcal{H}_{ac}(H) \subset \mathcal{H}_c(H), \tag{6.37}$$

we have only to show that

$$\mathcal{H}_c(H) \subset \mathcal{R}(W_\pm). \tag{6.38}$$

The definition of wave operators $W_{\pm}$ in (6.16) is known working only for short-range potentials satisfying (6.2) or (6.4). To extend (6.38) to include the long-range potentials satisfying (6.3), it is necessary to modify the definition (6.16). We here choose a modification of the form

$$W_{\pm}f = \lim_{t\to\pm\infty} e^{itH} J e^{-itH_0} f, \tag{6.39}$$

where J is an identification operator or time-independent modifier introduced in [19], and will be defined as a Fourier integral operator of the form

$$\begin{aligned} Jf(x) &= \text{Os-}\int\int e^{i(\varphi(x,\xi)-y\xi)} f(y) dy \widehat{d\xi} \\ &= c_m \int e^{i\varphi(x,\xi)} \hat{f}(\xi) d\xi. \end{aligned} \tag{6.40}$$

Here $\hat{f}$ is the Fourier transform of $f \in \mathcal{S}(R^m)$, $c_m = (2\pi)^{-m/2}$, and the phase function $\varphi(x,\xi)$ will be constructed as a solution of an eikonal equation

$$\frac{1}{2}|\nabla_x \varphi(x,\xi)|^2 + V_L(x) = \frac{1}{2}|\xi|^2$$

in the forward and backward regions, in which $x \in R^m$ and $\xi \in R^m$ are almost parallel and anti-parallel to each other, respectively. With $\varphi(x,\xi)$ being well-defined, we argue as follows: We appeal to the Cauchy criterion of convergence in proving the existence of $W_{\pm}$ and its asymptotic completeness: $\mathcal{R}(W_{\pm}) = \mathcal{H}_c(H)$, with using Theorem 3.2. For instance, the argument for the asymptotic completeness is as follows: We consider the case $t \to +\infty$ only. The other case is treated similarly. Noting that the vectors of the form $E_H(B)g$ with $g \in \mathcal{H}_c(H)$ and B being a Borel subset of the interval $(0,\infty)$ are dense in $\mathcal{H}_c(H)$, we evaluate the difference for $g \in \mathcal{H}_c(H)$ with $g = E_H(B)g$ $(B \subset (d^2/2, b),\ 0 < d,\ 0 < d^2/2 < b < \infty)$:

$$\begin{aligned} &(e^{itH_0} J^{-1} e^{-itH} - e^{it_mH_0} J^{-1} e^{-it_mH}) g \\ &\quad = e^{it_mH_0} (e^{i(t-t_m)H_0} J^{-1} e^{-i(t-t_m)H} - J^{-1}) e^{-it_mH} g. \end{aligned} \tag{6.41}$$

Here t_m is a sequence in Theorem 3.2 tending to infinity as $m \to \infty$ and J^{-1} is an inverse of J that exists for a suitable choice of φ. By virtue of Theorem 3.2, we can insert a factor $P_+F(|x| > R_m)$ between $(e^{i(t-t_m)H_0} J^{-1} e^{-i(t-t_m)H} - J^{-1})$ and $e^{-it_mH} g$ on the RHS of (6.41), where P_+ is a ψdo in (5.95) satisfying (5.100) with $\sigma = d^2/2$, and R_m is a suitable sequence tending to infinity as $m \to +\infty$ such that (6.13) holds with $R > 0$ replaced by $R_m > 0$. For the later use, we choose the symbol $p_+(\xi, y)$ of P_+ such that it satisfies in the forward region

$$|\partial_y^\alpha \partial_\xi^\beta p_+(\xi,y)| \le C_{\alpha\beta} \langle y \rangle^{-|\alpha|} \quad (\cos(y,\xi) \ge \theta + \rho). \tag{6.42}$$

for any multi-indices α, β, where the constant $C_{\alpha\beta} > 0$ is independent of (ξ, y). Since the role of P_+ is to restrict the support of the applied function to the forward sector, this restriction is not an essential one. We then estimate

$$\begin{aligned}
& \|(e^{i(t-t_m)H_0}J^{-1}e^{-i(t-t_m)H} - J^{-1})P_+F(|x| > R_m)\| \qquad (6.43)\\
= \ & \|e^{i(t-t_m)H_0}J^{-1}e^{-i(t-t_m)H}(J - e^{i(t-t_m)H}Je^{-i(t-t_m)H_0})J^{-1}P_+F(|x| > R_m)\| \\
\leq \ & C\left\| \int_0^{t-t_m} \frac{d}{ds}\left(e^{isH}Je^{-isH_0}\right) J^{-1}P_+F(|x| > R_m)ds \right\| \\
\leq \ & C\int_0^{t-t_m} \|(HJ - JH_0)e^{-isH_0}J^{-1}P_+F(|x| > R_m)\|ds.
\end{aligned}$$

Here by the eikonal equation, $T = HJ - JH_0$ decays in the forward direction to the order $\langle x\rangle^{-1-\delta}$ so that we can apply Theorems 5.6 and 5.7 to get

$$\begin{aligned}
& \|Te^{-isH_0}J^{-1}P_+F(|x| > R_m)\| \qquad (6.44)\\
& \leq \|Te^{-isH_0}J^{-1}P_+\langle x\rangle^{\delta/2}\|\|\langle x\rangle^{-\delta/2}F(|x| > R_m)\| \leq C\langle s\rangle^{-1-\delta/2}\langle R_m\rangle^{-\delta/2}.
\end{aligned}$$

These yield

$$\|(e^{itH_0}J^{-1}e^{-itH} - e^{it_mH_0}J^{-1}e^{-it_mH})g\| \to 0 \qquad (6.45)$$

as $t > t_m \to \infty$. Thus we have a Cauchy criterion for $t > s > t_m \to \infty$:

$$\begin{aligned}
& \|(e^{itH_0}J^{-1}e^{-itH} - e^{isH_0}J^{-1}e^{-isH})g\| \qquad (6.46)\\
& \quad \leq \|(e^{itH_0}J^{-1}e^{-itH} - e^{it_mH_0}J^{-1}e^{-it_mH})g\| \\
& \qquad + \|(e^{isH_0}J^{-1}e^{-isH} - e^{it_mH_0}J^{-1}e^{-it_mH})g\| \to 0.
\end{aligned}$$

This proves the existence of the limit

$$\Omega_+g = \lim_{t\to\infty} e^{itH_0}J^{-1}e^{-itH}g \qquad (6.47)$$

for $g \in \mathcal{H}_c(H)$. Combining this with the existence of W_+, we get for $g \in \mathcal{H}_c(H)$

$$g = W_+\Omega_+g \in \mathcal{R}(W_+), \qquad (6.48)$$

which will complete the proof of the asymptotic completeness.

To construct the phase function $\varphi(x, \xi)$ with the desired properties, we need to consider the classical orbits associated with the classical Hamiltonian:

$$H_\rho(t, x, \xi) = \frac{1}{2}|\xi|^2 + V_\rho(t, x). \qquad (6.49)$$

Here $0 < \rho < 1$ and

$$V_\rho(t, x) = V_L(x)\phi(\rho x)\phi\left(\frac{\langle \log \langle t\rangle\rangle}{\langle t\rangle}x\right), \qquad (6.50)$$

where $\phi(x)$ is a $C^\infty(R^m)$ function satisfying

$$\phi(x) = \begin{cases} 1 & |x| \geq 2 \\ 0 & |x| \leq 1 \end{cases} \qquad (6.51)$$

with $0 \le \phi(x) \le 1$. Then V_ρ satisfies

$$|\partial_x^\alpha V_\rho(t,x)| \le C_\alpha \rho^{\delta_0} \langle t\rangle^{-\ell} \langle x\rangle^{-m} \tag{6.52}$$

for $\ell, m \ge 0$ and $0 < \delta_0 < \delta$ such that $\delta_0 + \ell + m < |\alpha| + \delta$.

The corresponding classical orbit $(q,p)(t,s,y,\xi) = (q(t,s,y,\xi), p(t,s,y,\xi))$ is determined by the equation

$$\begin{cases} q(t,s) = y + \int_s^t p(\tau,s)d\tau, \\ p(t,s) = \xi - \int_s^t \nabla_x V_\rho(\tau, q(\tau,s))d\tau. \end{cases} \tag{6.53}$$

Letting $\delta_0, \delta_1 > 0$ be fixed as $0 < \delta_0 + \delta_1 < \delta$, we have the following estimates for $(q,p)(t,s,y,\xi)$, which are proved by solving the equation (6.53) by iteration:

Proposition 6.3 *There is a constant $C_\ell > 0$ ($\ell = 0,1,2,\cdots$) such that for all $(y,\xi) \in R^{2m}$, $\pm t \ge \pm s \ge 0$ and multi-index α:*

$$|p(s,t,y,\xi) - \xi| \le C_0 \rho^{\delta_0} \langle s\rangle^{-\delta_1}. \tag{6.54}$$
$$|\partial_y^\alpha[\nabla_y q(s,t,y,\xi) - I]| \le C_{|\alpha|} \rho^{\delta_0} \langle s\rangle^{-\delta_1}, \tag{6.55}$$
$$|\partial_y^\alpha[\nabla_y p(s,t,y,\xi)]| \le C_{|\alpha|} \rho^{\delta_0} \langle s\rangle^{-1-\delta_1}. \tag{6.56}$$
$$|\nabla_\xi q(t,s,y,\xi) - (t-s)I| \le C_0 \rho^{\delta_0} \langle s\rangle^{-\delta_1} |t-s|, \tag{6.57}$$
$$|\nabla_\xi p(t,s,y,\xi) - I| \le C_0 \rho^{\delta_0} \langle s\rangle^{-\delta_1}. \tag{6.58}$$
$$|\nabla_y q(t,s,y,\xi) - I| \le C_0 \rho^{\delta_0} \langle s\rangle^{-1-\delta_1} |t-s|, \tag{6.59}$$
$$|\nabla_y p(t,s,y,\xi)| \le C_0 \rho^{\delta_0} \langle s\rangle^{-1-\delta_1}. \tag{6.60}$$
$$\begin{aligned} &|\partial_\xi^\alpha[q(t,s,y,\xi) - y - (t-s)p(t,s,y,\xi)]| \\ &\qquad \le C_{|\alpha|} \rho^{\delta_0} \min(\langle t\rangle^{1-\delta_1}, |t-s|\langle s\rangle^{-\delta_1}). \end{aligned} \tag{6.61}$$

Further for any α, β satisfying $|\alpha+\beta| \ge 2$, there is a constant $C_{\alpha\beta} > 0$ such that

$$|\partial_y^\alpha \partial_\xi^\beta q(t,s,y,\xi)| \le C_{\alpha\beta} \rho^{\delta_0} |t-s| \langle s\rangle^{-\delta_1}, \tag{6.62}$$
$$|\partial_y^\alpha \partial_\xi^\beta p(t,s,y,\xi)| \le C_{\alpha\beta} \rho^{\delta_0} \langle s\rangle^{-\delta_1} \tag{6.63}$$

From this proposition, taking $\rho > 0$ so small that $C_0 \rho^{\delta_0} < 1/2$ we obtain the following

Proposition 6.4 *Take $\rho > 0$ so that $C_0 \rho^{\delta_0} < 1/2$. Then for $\pm t \ge \pm s \ge 0$ one can construct diffeomorphisms of R^m*

$$x \mapsto y(s,t,x,\xi) \tag{6.64}$$
$$\xi \mapsto \eta(t,s,x,\xi) \tag{6.65}$$

such that

$$\begin{cases} q(s,t,y(s,t,x,\xi),\xi) = x \\ p(t,s,x,\eta(t,s,x,\xi)) = \xi \end{cases}. \tag{6.66}$$

$y(s,t,x,\xi)$ and $\eta(t,s,x,\xi)$ are C^∞ in $(x,\xi)\in R^{2m}$ and their derivatives $\partial_x^\alpha\partial_\xi^\beta y$ and $\partial_x^\alpha\partial_\xi^\beta \eta$ are C^1 in (t,s,x,ξ). They satisfy the relation

$$\begin{cases} y(s,t,x,\xi) = q(t,s,x,\eta(t,s,x,\xi)) \\ \eta(t,s,x,\xi) = p(s,t,y(s,t,x,\xi),\xi) \end{cases} \tag{6.67}$$

and the estimates for any α,β

$$|\partial_x^\alpha\partial_\xi^\beta[\nabla_x y(s,t,x,\xi) - I]| \le C_{\alpha\beta}\rho^{\delta_0}\langle s\rangle^{-\delta_1}, \tag{6.68}$$

$$|\partial_x^\alpha\partial_\xi^\beta[\nabla_x \eta(t,s,x,\xi)]| \le C_{\alpha\beta}\rho^{\delta_0}\langle s\rangle^{-1-\delta_1}. \tag{6.69}$$

$$|\partial_\xi^\alpha[\eta(t,s,x,\xi) - \xi]| \le C_{\alpha}\rho^{\delta_0}\langle s\rangle^{-\delta_1} \tag{6.70}$$

$$|\partial_\xi^\alpha[y(s,t,x,\xi) - x - (t-s)\xi]| \le C_{\alpha}\rho^{\delta_0}\min(\langle t\rangle^{1-\delta_1}, |t-s|\langle s\rangle^{-\delta_1}). \tag{6.71}$$

Further for any $|\alpha+\beta|\ge 2$

$$|\partial_x^\alpha\partial_\xi^\beta \eta(t,s,x,\xi)| \le C_{\alpha\beta}\rho^{\delta_0}\langle s\rangle^{-\delta_1}, \tag{6.72}$$

$$|\partial_x^\alpha\partial_\xi^\beta y(s,t,x,\xi)| \le C_{\alpha\beta}\rho^{\delta_0}\langle t-s\rangle\langle s\rangle^{-\delta_1}. \tag{6.73}$$

Here the constants $C_\alpha, C_{\alpha\beta} > 0$ are independent of t,s,x,ξ

The following illustration would be helpful to understand the meaning of the diffeomorphisms $y(s,t,x,\xi)$ and $\eta(t,s,x,\xi)$: Let $U(t,s)$ be the map that assigns the point $(q,p)(t,s,x,\eta)$ to the initial data (x,η). Then

$$\begin{matrix} \text{time } s & & \text{time } t \\ \begin{pmatrix} x \\ \\ \eta(t,s,x,\xi) \end{pmatrix} & \overset{U(t,s)}{\longmapsto} & \begin{pmatrix} y(s,t,x,\xi) \\ \\ \xi \end{pmatrix} \end{matrix} \tag{6.74}$$

We now define $\phi(t,x,\xi)$ by

$$\phi(t,x,\xi) = u(t,x,\eta(t,0,x,\xi)), \tag{6.75}$$

where

$$u(t,x,\eta) = x\cdot\eta + \int_0^t (H_\rho - x\cdot\nabla_x H_\rho)(\tau, q(\tau,0,x,\eta), p(\tau,0,x,\eta))d\tau. \tag{6.76}$$

Then it is shown by a direct calculation that $\phi(t,x,\xi)$ satisfies the Hamilton-Jacobi equation

$$\begin{aligned} \partial_t\phi(t,x,\xi) &= \frac{1}{2}|\xi|^2 + V_\rho(t,\nabla_\xi\phi(t,x,\xi)), \\ \phi(0,x,\xi) &= x\cdot\xi, \end{aligned} \tag{6.77}$$

and the relation

$$\nabla_x\phi(t,x,\xi) = \eta(t,0,x,\xi), \tag{6.78}$$

$$\nabla_\xi\phi(t,x,\xi) = y(0,t,x,\xi). \tag{6.79}$$

We define for $(x,\xi) \in R^{2m}$

$$\phi_\pm(x,\xi) = \lim_{t\to\pm\infty}(\phi(t,x,\xi) - \phi(t,0,\xi)). \tag{6.80}$$

We will show the existence of the limits below. We set for $R, d > 0$ and $\sigma_0 \in (-1,1)$

$$\begin{aligned}\Gamma_\pm &= \Gamma_\pm(R,d,\sigma_0) \\ &= \{(x,\xi) \in R^{2m} \mid |x| \geq R, |\xi| \geq d, \pm\cos(x,\xi) \geq \pm\sigma_0\}.\end{aligned} \tag{6.81}$$

Proposition 6.5 *The limits (6.80) exist for all $(x,\xi) \in R^{2m}$ and define C^∞ functions of (x,ξ). The limit $\phi_\pm(x,\xi)$ satisfies the eikonal equation: For any $d > 0$ and $\sigma_0 \in (-1,1)$, there is a constant $R = R_d = R_{d\sigma_0} > 1$ such that for any $(x,\xi) \in \Gamma_\pm = \Gamma_\pm(R,d,\sigma_0)$, the following relation holds:*

$$\frac{1}{2}|\nabla_x\phi_\pm(x,\xi)|^2 + V_L(x) = \frac{1}{2}|\xi|^2. \tag{6.82}$$

Further for any α, β we have the estimate:

$$|\partial_x^\alpha\partial_\xi^\beta(\phi_\pm(x,\xi) - x\cdot\xi)| \leq C_{\alpha\beta}|\xi|^{-1}\langle x\rangle^{1-|\alpha|-\delta}, \tag{6.83}$$

where $C_{\alpha\beta} > 0$ is independent of $(x,\xi) \in \Gamma_\pm$.

Proof: We consider $\phi = \phi_+$ only. ϕ_- can be treated similarly. We first prove the existence of the limit (6.80) for $t \to +\infty$. To do so, setting

$$R(t,x,\xi) = \phi(t,x,\xi) - \phi(t,0,\xi), \tag{6.84}$$

we show the existence of the limits

$$\lim_{t\to\infty}\partial_x^\alpha\partial_\xi^\beta R(t,x,\xi) = \lim_{t\to\infty}\int_0^t \partial_x^\alpha\partial_\xi^\beta\partial_t R(\tau,x,\xi)d\tau + \partial_x^\alpha\partial_\xi^\beta(x\cdot\xi). \tag{6.85}$$

By Hamilton-Jacobi equation (6.77),

$$\begin{aligned}\partial_t R(t,x,\xi) &= \partial_t\phi(t,x,\xi) - \partial_t\phi(t,0,\xi) \\ &= V_\rho(t,\nabla_\xi\phi(t,x,\xi)) - V_\rho(t,\nabla_\xi\phi(t,0,\xi)) \\ &= (\nabla_\xi\phi(t,x,\xi) - \nabla_\xi\phi(t,0,\xi))\cdot a(t,x,\xi) \\ &= (y(0,t,x,\xi) - y(0,t,0,\xi))\cdot a((t,x,\xi) \\ &= \nabla_\xi R(t,x,\xi)\cdot a(t,x,\xi),\end{aligned} \tag{6.86}$$

where

$$a(t,x,\xi) = \int_0^1 (\nabla_x V_\rho)(t,\nabla_\xi\phi(t,0,\xi) + \theta\nabla_\xi R(t,x,\xi))d\theta, \tag{6.87}$$

$$\nabla_\xi R(t,x,\xi) = x\cdot\int_0^1 (\nabla_x y)(0,t,\theta x,\xi)d\theta. \tag{6.88}$$

By (6.68), we have for any α, β

$$|\partial_x^\alpha \partial_\xi^\beta \nabla_\xi R(t,x,\xi)| \le C_{\alpha\beta}\langle x\rangle. \tag{6.89}$$

By (6.71) and (6.79), for $|\beta| \ge 1$

$$|\partial_\xi^\beta \nabla_\xi \phi(t,0,\xi)| \le C_\beta |t|. \tag{6.90}$$

From this, (6.87), and (6.89), we have

$$|\partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)| \le C_{\alpha\beta}\langle t\rangle^{-1-\delta/2}\langle x\rangle^{|\alpha|+|\beta|}. \tag{6.91}$$

Thus by (6.86), (6.89) and (6.91), there exists the limit for any α, β

$$\lim_{t\to\infty} \partial_x^\alpha \partial_\xi^\beta R(t,x,\xi) = \int_0^\infty \partial_x^\alpha \partial_\xi^\beta \left(\nabla_\xi R(t,x,\xi)\cdot a(t,x,\xi)\right) dt + \partial_x^\alpha \partial_\xi^\beta (x\cdot\xi). \tag{6.92}$$

In particular, $\phi = \phi_+(x,\xi) = \lim_{t\to\infty} R(t,x,\xi)$ and $\eta(\infty,0,x,\xi) = \lim_{t\to\infty} \nabla_x \phi(t,x,\xi)$ exist and are C^∞ in (x,ξ).

Next we show (6.82). By the arguments above, the following limit exist:

$$\begin{aligned} \nabla_x \phi(x,\xi) &= \lim_{t\to\infty} \nabla_x \phi(t,x,\xi) = \lim_{t\to\infty} \eta(t,0,x,\xi) \\ &= \lim_{t\to\infty} p(0,t,y(0,t,x,\xi),\xi). \end{aligned} \tag{6.93}$$

Thus for a sufficiently large $|x|$ (i.e. for $|\rho x| \ge 2$) we have

$$\frac{1}{2}|\nabla_x \phi_+(x,\xi)|^2 + V_L(x) = \frac{1}{2}\lim_{t\to\infty} |p(0,t,y(0,t,x,\xi),\xi)|^2 + V_\rho(0,x). \tag{6.94}$$

Set for $0 \le s \le t < \infty$

$$f_t(s,y,\xi) = \frac{1}{2}|p(s,t,y,\xi)|^2 + V_\rho(s,q(s,t,y,\xi)). \tag{6.95}$$

Then by (6.53) we have

$$\begin{aligned} \frac{\partial f_t}{\partial s}(s,y,\xi) &= p(s,t,y,\xi)\cdot \partial_s p(s,t,y,\xi) \\ &\quad + (\nabla_x V_\rho)(s,q(s,t,y,\xi))\cdot \partial_s q(s,t,y,\xi) + \frac{\partial V_\rho}{\partial t}(s,q(s,t,y,\xi)) \\ &= \frac{\partial V_\rho}{\partial t}(s,q(s,t,y,\xi)). \end{aligned} \tag{6.96}$$

On the other hand we have from (6.66) and (6.67)

$$\begin{aligned} q(s,t,y(0,t,x,\xi),\xi) &= q(s,t,q(t,0,x,\eta(t,0,x,\xi)),\xi) \\ &= q(s,0,x,\eta(t,0,x,\xi)), \end{aligned} \tag{6.97}$$

$$\begin{aligned} p(s,t,y(0,t,x,\xi),\xi) &= p(s,t,q(t,0,x,\eta(t,0,x,\xi)),\xi) \\ &= p(s,0,x,\eta(t,0,x,\xi)). \end{aligned} \tag{6.98}$$

Now using Proposition 6.3, we have for $\cos(x,\xi) \geq \sigma_0$

$$\begin{aligned}|q(s,t,y(0,t,x,\xi),\xi)| &= |q(s,0,x,\eta(t,0,x,\xi))| \\ &\geq |x + sp(s,0,x,\eta(t,0,x,\xi))| - C_0\rho^{\delta_0}\langle s\rangle^{1-\delta_1} \\ &= |x + sp(s,t,y(0,t,x,\xi),\xi)| - C_0\rho^{\delta_0}\langle s\rangle^{1-\delta_1} \\ &\geq c(|x| + s|\xi|) - C_0\rho^{\delta_0}\langle s\rangle^{1-\delta_1} - C_0\rho^{\delta_0}\langle s\rangle^{1-\delta_1},\end{aligned} \tag{6.99}$$

where $c > 0$ is a constant independent of s,t,x,ξ. By $(x,\xi) \in \Gamma_+(R,d,\sigma_0)$, we have $|\xi| \geq d$, and from the definition (6.50) of $V_\rho(t,x)$

$$\operatorname{supp} \frac{\partial V_\rho}{\partial t}(s,x) \subset \{x | 1 \leq \langle \log \langle s\rangle\rangle |x|/\langle s\rangle \leq 2\}. \tag{6.100}$$

Thus there is a constant $S = S_{d,\sigma_0} > 1$ independent of t such that for any $s \in [S,t]$

$$\frac{\partial f_t}{\partial s}(s, y(0,t,x,\xi),\xi) = 0. \tag{6.101}$$

For $s \in [0,S]$, taking $R = R_S > 1$ large enough, we have for $|x| \geq R$ and $\cos(x,\xi) \geq \sigma_0$

$$\frac{\partial f_t}{\partial s}(s, y(0,t,x,\xi),\xi) = 0. \tag{6.102}$$

Therefore we have shown that for $(x,\xi) \in \Gamma_+(R,d,\sigma_0)$

$$f_t(s, y(0,t,x,\xi),\xi) = \text{constant for } 0 \leq s \leq t < \infty. \tag{6.103}$$

In particular we have

$$f_t(0, y(0,t,x,\xi),\xi) = f_t(t, y(0,t,x,\xi),\xi), \tag{6.104}$$

which means

$$\frac{1}{2}|p(0,t,y(0,t,x,\xi),\xi)|^2 + V_\rho(0,x) = \frac{1}{2}|\xi|^2 + V_\rho(t, y(0,t,x,\xi)). \tag{6.105}$$

Since $V_\rho(t,y) \to 0$ uniformly in $y \in R^m$ when $t \to \infty$ by (6.52), we have from this and (6.94)

$$\frac{1}{2}|\nabla_x \phi_+(x,\xi)|^2 + V_L(x) = \frac{1}{2}|\xi|^2 \quad \text{for} \quad (x,\xi) \in \Gamma_+(R,d,\sigma_0), \tag{6.106}$$

if $R > 1$ is sufficiently large.

We finally prove the estimates (6.83). We first consider the derivatives with respect to ξ:

$$\partial_\xi^\beta(\phi_+(x,\xi) - x\cdot\xi) = \int_0^\infty \partial_\xi^\beta \partial_t R(t,x,\xi)dt, \tag{6.107}$$

where as above $R(t,x,\xi) = \phi(t,x,\xi) - \phi(t,0,\xi)$. Set

$$\gamma(t,x,\xi) = y(0,t,x,\xi) - (x + t\xi) \tag{6.108}$$

for $(x,\xi)\in\Gamma_+(R,d,\sigma_0)$. Then by (6.71) we have for $\theta\in[0,1]$

$$\begin{aligned}|\nabla_\xi\phi(t,0,\xi)+\theta\nabla_\xi R(t,x,\xi)| &= |y(0,t,0,\xi)+\theta(y(0,t,x,\xi)-y(0,t,0,\xi))| \\ &= |t\xi+\gamma(t,0,\xi)+\theta(x+\gamma(t,x,\xi)-\gamma(t,0,\xi))| \\ &= |\theta x+t\xi+(1-\theta)\gamma(t,0,\xi)+\theta\gamma(t,x,\xi)| \\ &\geq c_0(\theta|x|+t|\xi|)-c_1\rho^{\delta_0}\min(\langle t\rangle^{1-\delta_1},|t|)\end{aligned} \tag{6.109}$$

for some constants $c_0,c_1>0$ independent of x,ξ,θ and $t\geq 0$. Thus there are constants $\rho\in(0,d)$ and $T=T_{d,\sigma_0}>0$ such that for all $t\geq T$ and $(x,\xi)\in\Gamma_+(R,d,\sigma_0)$

$$\langle\nabla_\xi\phi(t,0,\xi)+\theta\nabla_\xi R(t,x,\xi)\rangle^{-1}\leq C\langle\theta|x|+t|\xi|\rangle^{-1}. \tag{6.110}$$

Therefore $a(t,x,\xi)$ defined by (6.87) satisfies by (6.89) and (6.90)

$$|\partial_\xi^\beta a(t,x,\xi)|\leq C_\beta\int_0^1\langle\theta|x|+t|\xi|\rangle^{-1-\delta}d\theta. \tag{6.111}$$

Using (6.109), we see that (6.111) holds also for $t\in[0,T]$ if we take $\rho>0$ small enough. Therefore for all $(x,\xi)\in\Gamma_+(R,d,\sigma_0)$ we have from (6.86) and (6.89)

$$\begin{aligned}|\partial_\xi^\beta(\phi_+(x,\xi)-x\cdot\xi)| &\leq C_{T,\beta}\langle x\rangle\int_0^\infty\int_0^1\langle\theta|x|+t|\xi|\rangle^{-1-\delta}d\theta dt \\ &\leq C_{T,\beta}\langle x\rangle|\xi|^{-1}\int_0^1\langle\theta|x|\rangle^{-\delta}d\theta \\ &\leq C_{T,\beta}\langle x\rangle^{1-\delta}|\xi|^{-1}.\end{aligned} \tag{6.112}$$

We next consider

$$\begin{aligned}\nabla_x\phi_+(x,\xi)-\xi &= \lim_{t\to\infty}(\nabla_x\phi(t,x,\xi)-\xi) \\ &= \lim_{t\to\infty}(p(0,t,y(0,t,x,\xi),\xi)-\xi) \\ &= \lim_{t\to\infty}\int_0^t(\nabla_x V_\rho)(\tau,q(\tau,t,y(0,t,x,\xi),\xi))\,d\tau \\ &= \lim_{t\to\infty}\int_0^t(\nabla_x V_\rho)(\tau,q(\tau,0,x,\eta(t,0,x,\xi)))\,d\tau \\ &= \int_0^\infty(\nabla_x V_\rho)(\tau,q(\tau,0,x,\eta(\infty,0,x,\xi)))\,d\tau.\end{aligned} \tag{6.113}$$

By (6.54) and (6.61) of Proposition 6.3

$$\begin{aligned}|q(\tau,0,x,\eta(\infty,0,x,\xi))| &\geq |x+\tau p(\tau,0,x,\eta(\infty,0,x,\xi))|-C_0\rho^{\delta_0}\langle\tau\rangle^{1-\delta_1} \\ &\geq |x+\tau p(\tau,\infty,y(0,\infty,x,\xi),\xi)|-C_0\rho^{\delta_0}\langle\tau\rangle^{1-\delta_1} \\ &\geq |x+\tau\xi|-C_0\rho^{\delta_0}\langle\tau\rangle^{1-\delta_1}-C_0\rho^{\delta_0}\langle\tau\rangle^{1-\delta_1}.\end{aligned} \tag{6.114}$$

Thus taking $\rho>0$ sufficiently small and $R=R_{d,\sigma_0,\rho}>1$ sufficiently large, we have for $(x,\xi)\in\Gamma_+(R,d,\sigma_0)$

$$|q(\tau,0,x,\eta(\infty,0,x,\xi))|\geq c_0(|x|+\tau|\xi|) \tag{6.115}$$

for some constant $c_0 > 0$. Therefore we obtain

$$|\nabla_x \phi_+(x,\xi) - \xi| \le C \int_0^\infty \langle |x| + \tau|\xi|\rangle^{-1-\delta} d\tau \le C|\xi|^{-1}\langle x\rangle^{-\delta}. \tag{6.116}$$

For higher derivatives, the proof is similar. For example let us consider

$$\begin{aligned} \partial_\xi \partial_x \phi_+(x,\xi) - I &= \int_0^\infty \partial_\xi \{(\nabla_x V_\rho)(\tau, q(\tau,0,x,\eta(\infty,0,x,\xi)))\} d\tau \\ &= \int_0^\infty (\nabla_x \nabla_x V_\rho)(\tau, q(\tau,0,x,\eta(\infty,0,x,\xi)))\, \nabla_\xi q \cdot \nabla_\xi \eta d\tau, \end{aligned} \tag{6.117}$$

where we abbreviated $q = q(\tau,0,x,\eta(\infty,0,x,\xi))$ and $\eta = \eta(\infty,0,x,\xi)$. The RHS is bounded by a constant times

$$\int_0^\infty \langle |x| + \tau|\xi|\rangle^{-2-\delta} \langle \tau \rangle d\tau \le c|\xi|^{-1}\langle x\rangle^{-\delta} \tag{6.118}$$

for $(x,\xi) \in \Gamma_+(R,d,\sigma_0)$ by (6.57) and (6.70) of Propositions 6.3 and 6.4. Other estimates are proved similarly by using (6.57), (6.62), (6.70) and (6.72). □

Now let $-1 < \sigma_- < \sigma_+ < 1$ and take two functions $\psi_\pm(\sigma) \in C^\infty([-1,1])$ such that

$$0 \le \psi_\pm(\sigma) \le 1, \tag{6.119}$$

$$\psi_+(\sigma) = \begin{cases} 1, & \sigma_+ \le \sigma \le 1 \\ 0, & -1 \le \sigma \le \sigma_- \end{cases}, \tag{6.120}$$

$$\psi_-(\sigma) = 1 - \psi_+(\sigma) = \begin{cases} 0, & \sigma_+ \le \sigma \le 1 \\ 1, & -1 \le \sigma \le \sigma_- \end{cases}, \tag{6.121}$$

and set

$$\chi_\pm(x,\xi) = \psi_\pm(\cos(x,\xi)), \quad \left(\cos(x,\xi) = \frac{x\cdot\xi}{|x||\xi|}\right). \tag{6.122}$$

We then define the phase function $\varphi(x,\xi)$ by

$$\begin{aligned} &\varphi(x,\xi) \\ &= \{(\phi_+(x,\xi) - x\cdot\xi)\chi_+(x,\xi) + (\phi_-(x,\xi) - x\cdot\xi)\chi_-(x,\xi)\}\, \phi(2\xi/d)\phi(2x/R) + x\cdot\xi, \end{aligned} \tag{6.123}$$

where $\phi(x)$ is the function defined by (6.51). $\varphi(x,\xi)$ is a C^∞ function of $(x,\xi) \in R^{2m}$.

Noting that $\chi_+(x,\xi) + \chi_-(x,\xi) \equiv 1$ for $x \ne 0, \xi \ne 0$, we have proved the following theorem.

Theorem 6.6 *Let the notations be as above. Then for any $d > 0$ and $-1 < \sigma_- < \sigma_+ < 1$, there is $R = R_d = R_{d\sigma_\pm} > 1$ such that $R_d > 1$ increases as $d > 0$ decreases and the followings hold:*

i) *For $|\xi| \ge d$, $|x| \ge R$ and $\cos(x,\xi) \ge \sigma_+$ or $\cos(x,\xi) \le \sigma_-$*

$$\frac{1}{2}|\nabla_x \varphi(x,\xi)|^2 + V_L(x) = \frac{1}{2}|\xi|^2. \tag{6.124}$$

ii) *For any multi-indices* α, β *there is a constant* $C_{\alpha\beta} > 0$ *such that*

$$|\partial_x^\alpha \partial_\xi^\beta (\varphi(x,\xi) - x \cdot \xi)| \le C_{\alpha\beta} \langle x \rangle^{1-\delta-|\alpha|} \langle \xi \rangle^{-1}. \tag{6.125}$$

In particular for $|\alpha| \neq 0$*, we have for* $\delta_0, \delta_1 \ge 0$ *with* $\delta_0 + \delta_1 = \delta$

$$|\partial_x^\alpha \partial_\xi^\beta (\varphi(x,\xi) - x \cdot \xi)| \le C_{\alpha\beta} R^{-\delta_0} \langle x \rangle^{1-\delta_1-|\alpha|} \langle \xi \rangle^{-1}. \tag{6.126}$$

iii) *Set*

$$\begin{aligned} a(x,\xi) &= e^{-i\varphi(x,\xi)} \left(-\frac{1}{2}\Delta + V_L(x) - \frac{1}{2}|\xi|^2 \right) e^{i\varphi(x,\xi)} \qquad (6.127) \\ &= \frac{1}{2}|\nabla_x \varphi(x,\xi)|^2 + V_L(x) - \frac{1}{2}|\xi|^2 - \frac{i}{2}\Delta_x \varphi(x,\xi). \end{aligned}$$

Then $a(x,\xi)$ *satisfies for* $|\xi| \ge d$, $|x| \ge R$ *and any* α, β

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le \begin{cases} C_{\alpha\beta} \langle x \rangle^{-1-\delta-|\alpha|} \langle \xi \rangle^{-1}, & \cos(x,\xi) \in [-1, \sigma_-] \cup [\sigma_+, 1] \\ C_{\alpha\beta} \langle x \rangle^{-\delta-|\alpha|}, & \cos(x,\xi) \in [\sigma_-, \sigma_+] \end{cases} \tag{6.128}$$

We can now define the identification operator J for $f \in \mathcal{S}(R^m)$:

$$\begin{aligned} Jf(x) &= \text{Os-} \int\int e^{i(\varphi(x,\xi) - y\xi)} f(y) dy d\hat{\xi} \qquad (6.129) \\ &= c_m \int e^{i\varphi(x,\xi)} \hat{f}(\xi) d\xi, \end{aligned}$$

where $c_m = (2\pi)^{-m/2}$. We remark that this definition of J depends on the choice of the constants $d > 0$, $R = R_{d\sigma_\pm} > 1$ and σ_-, σ_+ with $-1 < \sigma_- < \sigma_+ < 1$ by the definition of $\varphi(x,\xi)$ in (6.123). But when two phase functions $\varphi_{d_1,R_{d_1}}$ and $\varphi_{d_2,R_{d_2}}$ corresponding to the constants $d_2 > d_1 > 0$ with the same $\sigma_\pm$ are given, they coincide with each other on the common region $\Gamma_\pm(R_{d_1}, d_2, \sigma_\pm)$, since the limits (6.80) exist for all $(x,\xi) \in R^{2m}$. In the following we fix a pair (σ_-, σ_+) with $-1 < \sigma_- < \sigma_+ < 1$ but vary the constants $d > 0$ and $R = R_d > 1$ in accordance with the context, and write $J = J_d$ when necessary to denote J with the phase function satisfying (6.124) for $|\xi| \ge d$. The function $a(x,\xi)$ in iii) of the theorem satisfies for $f \in \mathcal{S}$

$$\begin{aligned} Tf(x) &= (HJ - JH_0) f(x) \qquad (6.130) \\ &= \int\int e^{i(\varphi(x,\xi) - y\xi)} \{a(x,\xi) + V_S(x)\} f(y) dy d\hat{\xi} \\ &= c_m \int e^{i\varphi(x,\xi)} \{a(x,\xi) + V_S(x)\} \hat{f}(\xi) d\xi. \end{aligned}$$

Thus Theorem 6.6 tells that T satisfies the properties required for our arguments of the asymptotic completeness that we have stated at the beginning of this section.

To see that the wave operators (6.39) define bounded operators, we prove that J is bounded. To this end, since J is densely defined in $\mathcal{H} = L^2(R^m)$, it suffices to show that the adjoint operator J^* is bounded. It is defined for $f \in \mathcal{S}$ by

$$J^* f(x) = \text{Os-}\int\int e^{i(x\xi - \varphi(y,\xi))} f(y) dy d\widehat{\xi}. \tag{6.131}$$

By the inequality (see the proof of Theorem 5.4)

$$\|J^*\|^2 \leq \|J^{**} J^*\|, \tag{6.132}$$

we have only to prove the boundedness of the operator

$$J^{**} J^* f(x) = \text{Os-}\int\int e^{i(\varphi(x,\xi) - \varphi(y,\xi))} f(y) dy d\widehat{\xi}. \tag{6.133}$$

We compute

$$\begin{aligned} \varphi(x,\xi) - \varphi(y,\xi) &= (x-y) \cdot \int_0^1 \nabla_x \varphi(y + \theta(x-y), \xi) d\theta \\ &=: (x-y) \cdot \widetilde{\nabla}_x \varphi(x,\xi,y). \end{aligned} \tag{6.134}$$

By (6.126), we have

$$|\nabla_\xi \widetilde{\nabla}_x \varphi(x,\xi,y) - I_m| \leq C R^{-\delta_0}, \tag{6.135}$$

where I_m is the unit matrix of order m. Thus switching to a larger $R = R_d \geq 1$ if necessary, we can make the RHS of (6.135) less than 1/2. Then if we set $G_\eta(\xi) = \xi - \widetilde{\nabla}_x \varphi(x,\xi,y) + \eta$, we have

$$\begin{aligned} |G_\eta(\xi) - G_\eta(\xi')| &= |(I_m - \nabla_\xi \widetilde{\nabla}_x \varphi(x,\xi,y))(\xi - \xi')| \\ &\leq \frac{1}{2} |\xi - \xi'|. \end{aligned} \tag{6.136}$$

Therefore $G_\eta(\xi)$ is a contraction mapping from R^m into R^m, and there is a unique fixed point ξ of $G_\eta(\xi)$ for any $\eta \in R^m$:

$$G_\eta(\xi) = \xi, \quad \text{i.e.} \quad \eta = \widetilde{\nabla}_x \varphi(x,\xi,y). \tag{6.137}$$

Thus the inverse $\widetilde{\nabla}_x \varphi^{-1}(x,\eta,y)$ of the mapping

$$R^m \ni \xi \mapsto \eta = \widetilde{\nabla}_x \varphi(x,\xi,y) \in R^m \tag{6.138}$$

exists and defines a diffeomorphism of R^m. We make a change of variable in (6.133) by this transformation. Then denoting the Jacobian of $\widetilde{\nabla}_x \varphi^{-1}(x,\eta,y)$ by $J(x,\eta,y) = |\det(\nabla_\eta \widetilde{\nabla}_x \varphi^{-1}(x,\eta,y))|$, we obtain

$$J^{**} J^* f(x) = \text{Os-}\int\int e^{i(x-y)\eta} J(x,\eta,y) f(y) dy d\widehat{\eta}. \tag{6.139}$$

Since (6.125) implies the estimates

$$|\partial_x^\alpha \partial_\eta^\beta \partial_y^\gamma J(x,\eta,y)| \le C_{\alpha\beta\gamma} \tag{6.140}$$

for all α, β, γ with constants $C_{\alpha\beta\gamma} > 0$ independent of (x, η, y), we have from Theorem 5.4 that $J^{**}J^*$ is a bounded operator on $\mathcal{H} = L^2(R^m)$. This proves that J is extended to a bounded operator from $\mathcal{H}$ into itself and $J = J^{**}$.

We next prove that J has a bounded inverse. From (6.139), we have

$$(I - JJ^*)f(x) = \text{Os-}\int\int e^{i(x-y)\eta}(1 - J(x,\eta,y))f(y)dyd\widehat{\eta}. \tag{6.141}$$

Let $m_0 = 2[m/2 + 1]$ as in section 5.3 and $C_0 > 0$ be the constant in (5.91). Then we have by (6.126)

$$\sup_{|\alpha+\beta+\gamma|\le 3m_0} \sup_{x,\eta,y} |\partial_x^\alpha \partial_\eta^\beta \partial_y^\gamma (1 - J(x,\eta,y))| \le C_{m_0} R^{-\delta_0} < \frac{1}{2C_0} \tag{6.142}$$

by taking $R > 1$ large enough. Then we have from Theorem 5.4 that

$$\|I - JJ^*\| \le \frac{1}{2}. \tag{6.143}$$

Thus JJ^* is invertible with the inverse

$$(JJ^*)^{-1} = (I - (I - JJ^*))^{-1} = \sum_{j=0}^{\infty}(I - JJ^*)^j, \tag{6.144}$$

whose RHS converges in operator norm by (6.143). This implies that the range $\mathcal{R}(J)$ equals $\mathcal{H}$ and J^* is one-to-one. Furthermore, (6.142) implies that the symbol $r(x,\eta,y) = 1 - J(x,\eta,y)$ of the ψdo $I - JJ^*$ is small so that the series of symbols

$$q_1(x,\eta,y) = \sum_{j=0}^{\infty} r_j(x,\eta,y) \tag{6.145}$$

converges in the Fréche space $S_{0,0}$ of symbols $p(x,\eta,y)$ whose semi-norms are

$$|p|_\ell = \sup_{|\alpha+\beta+\gamma|\le\ell} \sup_{x,\eta,y} |\partial_x^\alpha \partial_\eta^\beta \partial_y^\gamma p(x,\eta,y)| < \infty \quad (\ell = 0,1,2,\cdots). \tag{6.146}$$

Here in (6.145), $r_j(x,\eta,y)$ is the symbol of the ψdo $(I - JJ^*)^j$ $(j = 0,1,2,\cdots)$. To see that (6.145) converges in $S_{0,0}$, we note that the inequality (5.90) stated before Theorem 5.4 implies

$$|r_j|_\ell \le (C_0)^j \sum_{|\ell_1+\cdots+\ell_j|\le\ell} \prod_{k=1}^{j} |r|_{3m_0+|\ell_k|} \le (C_0)^j (|r|_{3m_0})^{j-\ell} (|r|_{3m_0+\ell})^\ell \left(\sum_{|\ell_1+\cdots+\ell_j|\le\ell} 1 \right) \tag{6.147}$$

where $r(x,\eta,y) = r_1(x,\eta,y) = (1 - J)(x,\eta,y)$. Since

$$\sum_{|\ell_1+\cdots+\ell_j|\le\ell} 1 = \sum_{k=0}^{\ell} \begin{pmatrix} 3j+k-1 \\ k \end{pmatrix} \le C_\ell j^\ell \tag{6.148}$$

for some constant $C_\ell > 0$ independent of $j = 0, 1, 2, \cdots$ (recall that ℓ_i's are 3-dimensional multi-indices in (5.90)), we have

$$|r_j|_\ell \le C_\ell j^\ell (C_0|r|_{3m_0})^{j-\ell}(C_0|r|_{3m_0+\ell})^\ell. \tag{6.149}$$

Thus (6.142):

$$C_0|r|_{3m_0} < \frac{1}{2}, \tag{6.150}$$

implies the convergence of the series (6.145) in the symbol space $S_{0,0}$, and we have from (6.144) with $Q_1 = q_1(X, D_x, X')$

$$Q_1(JJ^*) = (JJ^*)Q_1 = I. \tag{6.151}$$

We next consider for $g \in \mathcal{S}$

$$\mathcal{F}J^*J\mathcal{F}^{-1}g(\xi) = \text{Os-}\int\int e^{-i(\varphi(y,\xi)-\varphi(y,\eta))}g(\eta)dyd\widehat{\eta}. \tag{6.152}$$

Similarly to (6.134), we write

$$\varphi(y,\xi) - \varphi(y,\eta) = (\xi - \eta)\cdot\widetilde{\nabla}_\xi\varphi(\xi, y, \eta), \tag{6.153}$$

where

$$\widetilde{\nabla}_\xi\varphi(\xi, y, \eta) = \int_0^1 \nabla_\xi\varphi(y, \eta + \theta(\xi - \eta))d\theta. \tag{6.154}$$

Then noting that the inequality similar to (6.135) holds also for $\widetilde{\nabla}_\xi\varphi(\xi, y, \eta)$, we make a change of variable:

$$z = \widetilde{\nabla}_\xi\varphi(\xi, y, \eta), \tag{6.155}$$

and obtain

$$\mathcal{F}J^*J\mathcal{F}^{-1}g(\xi) = \text{Os-}\int\int e^{-i(\xi-\eta)z}J(\xi, z, \eta)g(\eta)dzd\widehat{\eta}, \tag{6.156}$$

Here $J(\xi, z, \eta) = |\det(\nabla_y\widetilde{\nabla}_\xi\varphi^{-1}(\xi, z, \eta))|$ is a Jacobian, which belongs to $S_{0,0}$. Arguing similarly to the case of JJ^*, we can now construct a ψdo $Q_2 = q_2(X, D_x, X')$ that satisfies $q_2 \in S_{0,0}$ and

$$Q_2(J^*J) = (J^*J)Q_2 = I. \tag{6.157}$$

(6.157) and (6.151) show that J has an inverse

$$J^{-1} : \mathcal{H} \to \mathcal{H} \tag{6.158}$$

that is expressed as

$$J^{-1} = J^*Q_1 = Q_2J^* = q_2(X, D_x, X')J^*. \tag{6.159}$$

Since J^* is bounded as we have seen, and $Q_2 = q_2(X, D_x, X')$ is bounded on $\mathcal{H}$ by $q_2 \in S_{0,0}$ and Theorem 5.4, $J^{-1} = Q_2 J^*$ is also a bounded operator on $\mathcal{H}$.

We are now prepared to prove the existence and asymptotic completeness of the wave operators (6.39). Before going to the proof, we remark that the definition (6.39) should be understood as follows with taking into account the dependence of the identification operator $J = J_d$ on $d > 0$. Namely for $f \in \mathcal{F}^{-1}C_0^\infty(R^m)$, the support of whose Fourier transform is contained in a set $\Sigma(d) := \{\xi |\ |\xi| \ge d\}$, we define $W_\pm f$ by (6.39) with taking $J = J_d$. Noting that the phase function $\varphi(x, \xi)$ of J_d is equal to

$$\varphi(x,\xi) = \phi_+(x,\xi)\chi_+(x,\xi) + \phi_-(x,\xi)\chi_-(x,\xi) \tag{6.160}$$

in $\Sigma(d) \cap \{x |\ |x| \ge R\}$, we have a definition of $W_\pm$ independent of $d > 0$ and $R = R_d > 1$ by extending this $W_\pm$ to the whole space $\mathcal{H}$ by preserving the boundedness.

Since the proof of the existence of $W_\pm$ is quite similar to and simpler than that of the asymptotic completeness, we only prove the latter. We have already stated the outline of the proof of the asymptotic completeness at the beginning of this section. There what should be noted is that we prove the existence of the limit (6.47) for $g = E_H(B)g$, where the Borel set B is a subset of $(d^2/2, b)$ for some $d > 0$ with $0 < d^2/2 < b < \infty$. Then by (6.14), the energy restriction $E_H(B)$ is translated into the restriction $E_0(B)$ on the state $e^{-it_mH}g = E_H(B)e^{-it_mH}g$ as $m \to \infty$. Thus asymptotically as $t_m \to \infty$, we can assume that the ξ-support of the symbol $p_+(\xi, y)$ of P_+ satisfies (5.100) with $\sigma = d/2$ on the state $e^{-it_mH}g$. Then we can let $J = J_d$ in (6.41), and take P_+ thereafter so that its symbol $p_+(\xi, y)$ satisfies (5.97), (6.42), and (5.100) with $\sigma = d/2$. What remains to be proved is then the estimation of the following factor in (6.44):

$$\|Te^{-isH_0}J^{-1}P_+\langle x\rangle^{\delta/2}\|. \tag{6.161}$$

By (6.159) we have

$$Te^{-isH_0}J^{-1}P_+\langle x\rangle^{\delta/2} = Te^{-isH_0}Q_2J^*P_+\langle x\rangle^{\delta/2}. \tag{6.162}$$

By (6.15), the constant $\theta + \rho$ in (5.97) can be taken arbitrarily as far as $-1 < \theta + \rho < 1$. We here take $\theta + \rho = \sigma_+ + \rho < 1$ with $\rho > 0$, where $\sigma_+ \in (-1, 1)$ is the number specified in Theorem 6.6. We write

$$J^*P_+\langle x\rangle^{\delta/2} = J^*P_+\langle x\rangle^{\delta/2}JJ^{-1}. \tag{6.163}$$

The last factor J^{-1} is bounded and can be omitted in the estimation. So we have to estimate

$$Te^{-isH_0}Q_2J^*P_+\langle x\rangle^{\delta/2}J. \tag{6.164}$$

By a calculation

$$\mathcal{F}J^*P_+\langle x\rangle^{\delta/2}J\mathcal{F}^{-1}\hat{f}(\xi) = \text{Os-}\int\int e^{-i(\varphi(y,\xi)-\varphi(y,\eta))}r_+(y,\eta)\hat{f}(\eta)dy\widehat{d\eta}. \tag{6.165}$$

Here the symbol $r_+(y,\eta)$ is given by

$$r_+(y,\eta) = \text{Os-}\int\int e^{i(y-z)\zeta} p_+(\zeta + \widetilde{\nabla}_x\varphi(y,\eta,z), z)\langle z\rangle^{\delta/2} dz d\widehat{\zeta} \tag{6.166}$$

and satisfies by (6.42)

$$|r_+|_\ell^{(-1,\delta/2)} := \sup_{|\alpha+\beta|\le\ell}\sup_{y,\eta} |\langle y\rangle^{-\delta/2}\langle y\rangle^{|\alpha|}\partial_y^\alpha\partial_\eta^\beta r_+(y,\eta)| < \infty, \quad (\ell = 0,1,2,\cdots). \tag{6.167}$$

We denote by $S_{-1,\delta/2}$ the space of symbols that satisfy this estimate. As in (6.153), we write

$$\varphi(y,\xi) - \varphi(y,\eta) = (\xi-\eta)\cdot\widetilde{\nabla}_\xi\varphi(\xi,y,\eta), \tag{6.168}$$

where

$$\widetilde{\nabla}_\xi\varphi(\xi,y,\eta) = \int_0^1 \nabla_\xi\varphi(y,\eta+\theta(\xi-\eta))d\theta. \tag{6.169}$$

We then make a change of variable:

$$z = \widetilde{\nabla}_\xi\varphi(\xi,y,\eta). \tag{6.170}$$

Letting $J(\xi,z,\eta)$ be the Jacobian of the inverse mapping $\widetilde{\nabla}_\xi\varphi^{-1}(\xi,z,\eta)$, we write (6.165) as

$$\mathcal{F}J^*P_+\langle x\rangle^{\delta/2}J\mathcal{F}^{-1}\hat{f}(\xi) = \text{Os-}\int\int e^{-i(\xi-\eta)z}\tilde{r}_+(\xi,z,\eta)\hat{f}(\eta)dz d\widehat{\eta}, \tag{6.171}$$

where

$$\tilde{r}_+(\xi,z,\eta) = r_+(\widetilde{\nabla}_\xi\varphi^{-1}(\xi,z,\eta),\eta)J(\xi,z,\eta). \tag{6.172}$$

By our additional assumption (6.42) on $p_+(\xi,y)$ stated at the beginning of this section and by the estimates (6.125) for $\varphi(x,\xi)$, $\tilde{r}_+(\xi,z,\eta)$ belongs to the symbol space $S_{0,\delta/2}$ whose semi-norms are

$$|\tilde{r}_+|_\ell^{(0,\delta/2)} = \sup_{|\alpha+\beta+\gamma|\le\ell}\sup_{\xi,z,\eta} |\langle z\rangle^{-\delta/2}\partial_\xi^\alpha\partial_z^\beta\partial_\eta^\gamma\tilde{r}_+(\xi,z,\eta)| < \infty, \quad (\ell = 0,1,2,\cdots). \tag{6.173}$$

Further, by the property (5.97) of p_+, $\tilde{r}_{+L}$ defined as in Proposition 5.3 from $\tilde{r}_+$ satisfies

$$|\partial_\xi^\alpha\partial_z^\beta\tilde{r}_{+L}(\xi,z)| \le C_{\ell\alpha\beta}\langle z\rangle^{-\ell} \quad (\cos(z,\xi) < \sigma_+ + \rho/2) \tag{6.174}$$

for any $\ell = 0,1,2,\cdots$. We return to the configuration space by Fourier inversion and obtain

$$\begin{aligned} J^*P_+\langle x\rangle^{\delta/2}Jf(x) &= \text{Os-}\int\int e^{i(x-z)\xi}s(\xi,z)\langle z\rangle^{\delta/2}f(z)dz d\widehat{\xi} \\ &= s(D_x,X')\langle x\rangle^{\delta/2}f(x), \end{aligned} \tag{6.175}$$

where $s(\xi, z) \in S_{0,0}$ satisfies (6.174) with $\sigma_+ + \rho/2$ replaced by $\sigma_+ + \rho/3$ by virtue of (6.174), (6.172), (6.166), supp $p_+ \subset \{(\xi, y) | \ |\xi| \geq d/2, |y| \geq d/2\}$, and $J(\xi, z, \eta) \in S_{0,0}$. Thus the problem is reduced to the estimation of

$$\|Te^{-isH_0}Q_2 s(D_x, X')\langle x\rangle^{\delta/2} f\| \leq \|(J^*)^{-1}\| \|J^*Te^{-isH_0}Q_2 s(D_x, X')\langle x\rangle^{\delta/2} f\|. \quad (6.176)$$

Arguing as above and reducing J^*T to a ψdo, and recalling the estimate (6.128) for the symbol of T in Theorem 6.6, we can bound the RHS by a constant times $\langle s\rangle^{-1-\delta/2}$ by applying Theorems 5.6 and 5.7, as announced in (6.44). The proof of the asymptotic completeness is complete:

Theorem 6.7 *Let (6.2) and (6.3) be satisfied. Let J be defined as above. Then the wave operators*

$$W_\pm = s\text{-}\lim_{t\to\pm\infty} e^{itH} J e^{-itH_0} \quad (6.177)$$

exist and define bounded operators on $\mathcal{H}$, and the asymptotic completeness holds:

$$\mathcal{R}(W_\pm) = \mathcal{H}_{ac}(H) = \mathcal{H}_c(H). \quad (6.178)$$

Further $W_\pm$ intertwine H and H_0: for any Borel set B in R^1

$$E_H(B)W_\pm = W_\pm E_0(B). \quad (6.179)$$

We reformulate our problem as follows.

Let an inner product $(\cdot,\cdot)_J$ in $\mathcal{H}$ be defined by

$$(f, g)_J = (Jf, Jg)_{\mathcal{H}}. \quad (6.180)$$

It is clear that the space $\mathcal{H}$ becomes a Hilbert space $\mathcal{H}_J$ with this inner product. Further as J and J^{-1} are bounded operators from $\mathcal{H}$ onto $\mathcal{H}$, the two inner products $(\cdot,\cdot)_{\mathcal{H}}$ and $(\cdot,\cdot)_J$ are equivalent.

We consider an operator

$$H_J = J^{-1}HJ \quad (6.181)$$

in $\mathcal{H}_J$. Then it is clear that H_J is a selfadjoint operator in $\mathcal{H}_J$. And we have proved in the above that the limit

$$W_\pm^J f = \lim_{t\to\pm\infty} e^{itH_J} e^{-itH_0} f \quad (6.182)$$

exists for all $f \in \mathcal{H}$, and that it is asymptotic complete

$$\mathcal{R}(W_\pm^J) = \mathcal{H}_c(H) = \mathcal{H}_c(H_J), \quad (6.183)$$

where $\mathcal{H}_c(H_J)$ is understood to be defined in the space $\mathcal{H}_J$.

Those mean that we can regard that H_J is a Hamiltonian obtained from H_0 with being perturbed by a sort of "short-range" perturbation $V_J = H_J - H_0$ between the spaces $\mathcal{H}$ and $\mathcal{H}_J$:

$$H_J = H_0 + V_J. \quad (6.184)$$

Chapter 7

Many-Body Hamiltonian

We begin with some repetition of notations given in chapter 2 and section 3.2 with modifications to the present situation.

7.1 Preliminaries

We consider the Schrödinger operator defined in $L^2(R^{\nu N})$ ($\nu \geq 1$, $N \geq 2$)

$$H = H_0 + V, \quad H_0 = -\sum_{i=1}^{N} \frac{\hbar^2}{2m_i} \frac{\partial^2}{\partial r_i^2}. \tag{7.1}$$

Here

$$V = \sum_{\alpha} V_\alpha(x_\alpha), \tag{7.2}$$

where $x_\alpha = r_i - r_j$, $r_i = (r_{i1}, \cdots, r_{i\nu}) \in R^\nu$ is the position vector of the i-th particle, $\frac{\partial}{\partial r_i} = \left(\frac{\partial}{\partial r_{i1}}, \cdots, \frac{\partial}{\partial r_{i\nu}}\right)$, $\frac{\partial^2}{\partial r_i^2} = \sum_{j=1}^{\nu} \frac{\partial^2}{\partial r_{ij}^2} = \Delta_{r_i}$, $m_i > 0$ is the mass of the i-th particle, and $\alpha = \{i, j\}$ is a pair with $1 \leq i < j \leq N$. Our assumption on the decay rate of the pair potentials $V_\alpha(x_\alpha)$ is as follows.

Assumption 7.1 *$V_\alpha(x)$ ($x \in R^\nu$) is split into a sum of a real-valued C^∞ function $V_\alpha^L(x)$ and a real-valued measurable function $V_\alpha^S(x)$ of $x \in R^\nu$ satisfying the following conditions: There are real numbers ϵ and ϵ_1 with $0 < \epsilon, \epsilon_1 < 1$ such that for all multi-indices β*

$$|\partial_x^\beta V_\alpha^L(x)| \leq C_\beta \langle x \rangle^{-|\beta|-\epsilon} \tag{7.3}$$

with some constants $C_\beta > 0$ independent of $x \in R^\nu$, and

$$\langle x \rangle^{1+\epsilon_1} V_\alpha^S(x)(-\Delta_x + 1)^{-1} \text{ is a bounded operator in } L^2(R^\nu). \tag{7.4}$$

Here Δ_x is a Laplacian with respect to x, and $\langle x \rangle$ is a C^∞ function of x such that $\langle x \rangle = |x|$ for $|x| \geq 1$ and $\geq \frac{1}{2}$ for $|x| < 1$.

We can adopt weaker conditions on the differentiability and decay rate for higher derivatives of the long-range part $V_\alpha^L(x)$, but for later convenience of exposition, we adopt this form in the present paper.

The free part H_0 of H in (7.1) has various forms in accordance with our choice of coordinate systems. We use the so-called Jacobi coordinates. The center of mass of our N-particle system is

$$X_C = \frac{m_1 r_1 + \cdots + m_N r_N}{m_1 + \cdots + m_N},$$

and the Jacobi coordinates are defined by

$$x_i = r_{i+1} - \frac{m_1 r_1 + \cdots + m_i r_i}{m_1 + \cdots + m_i}, \quad i = 1, 2, \cdots, N-1. \tag{7.5}$$

Accordingly the corresponding canonically conjugate momentum operators are defined by

$$P_C = \frac{\hbar}{i}\frac{\partial}{\partial X_C}, \quad p_i = \frac{\hbar}{i}\frac{\partial}{\partial x_i}.$$

Using these new X_C, P_C, x_i, p_i, we can rewrite H_0 in (7.1) as

$$H_0 = \tilde{H}_0 + H_C. \tag{7.6}$$

Here

$$\tilde{H}_0 = \sum_{i=1}^{N-1} \frac{1}{2\mu_i} p_i^2 = -\sum_{i=1}^{N-1} \frac{\hbar^2}{2\mu_i} \Delta_{x_i}, \quad H_C = \frac{1}{\sum_{j=1}^N m_j} P_C^2,$$

where $\mu_i > 0$ is the reduced mass defined by the relation:

$$\frac{1}{\mu_i} = \frac{1}{m_{i+1}} + \frac{1}{m_1 + \cdots + m_i}.$$

The new coordinates give a decomposition $L^2(R^{\nu N}) = L^2(R^\nu) \otimes L^2(R^n)$ with $n = \nu(N-1)$ and in this decomposition, H is decomposed

$$H = H_C \otimes I + I \otimes \tilde{H}, \quad \tilde{H} = \tilde{H}_0 + V.$$

H_C is a Laplacian, so we consider $\tilde{H}$ in the Hilbert space $\mathcal{H} = L^2(R^n) = L^2(R^{\nu(N-1)})$. We write this $\tilde{H}$ as H in the followings:

$$H = H_0 + V = \sum_{i=1}^{N-1} \frac{1}{2\mu_i} p_i^2 + \sum_\alpha V_\alpha(x_\alpha) = -\sum_{i=1}^{N-1} \frac{\hbar^2}{2\mu_i} \Delta_{x_i} + \sum_\alpha V_\alpha(x_\alpha). \tag{7.7}$$

This means that we consider the Hamiltonian H in (7.1) restricted to the subspace of $R^{\nu N}$:

$$(m_1 + \cdots + m_N) X_C = m_1 r_1 + \cdots + m_N r_N = 0. \tag{7.8}$$

We equip this subspace with the inner product:

$$\langle x, y \rangle = \sum_{i=1}^{N-1} \mu_i x_i \cdot y_i, \tag{7.9}$$

where $\cdot$ denotes the Euclidean scalar product. With respect to this inner product, the changes of variables between Jacobi coordinates in (7.5) are realized by orthogonal transformations on the space R^n defined by (7.8), while μ_i and x_i depend on the order of the construction of the Jacobi coordinates in (7.5). If we define velocity operator v by

$$v = (v_1, \cdots, v_{N-1}) = (\mu_1^{-1} p_1, \cdots, \mu_{N-1}^{-1} p_{N-1}),$$

we can write using the inner product above

$$H_0 = \frac{1}{2}\langle v, v\rangle. \tag{7.10}$$

Next we introduce clustered Jacobi coordinate. Let $a = \{C_1, \cdots, C_k\}$ be a disjoint decomposition of the set $\{1, 2, \cdots, N\}$: $C_j \neq \emptyset$ $(j = 1, 2, \cdots, k)$, $\cup_{j=1}^k C_j = \{1, 2, \cdots, N\}$ with $C_i \cap C_j = \emptyset$ when $i \neq j$. We denote the number of elements of a set S by $|S|$. Then $|a| = k$ in the present case, and we call a a cluster decomposition with $|a|$ clusters $C_1, \cdots, C_{|a|}$. A clustered Jacobi coordinate $x = (x_a, x^a)$ associated with a cluster decomposition $a = \{C_1, \cdots, C_k\}$ is obtained by first choosing a Jacobi coordinate

$$x^{(C_\ell)} = (x_1^{(C_\ell)}, \cdots, x_{|C_\ell|-1}^{(C_\ell)}) \in R^{\nu(|C_\ell|-1)} \quad (\ell = 1, 2, \cdots, k)$$

for the $|C_\ell|$ particles in the cluster C_ℓ and then by choosing an intercluster Jacobi coordinate

$$x_a = (x_1, \cdots, x_{k-1}) \in R^{\nu(k-1)}$$

for the center of mass of the k clusters $C_1, \cdots, C_k$. Then $x^a = (x^{(C_1)}, \cdots, x^{(C_k)}) \in R^{\nu(N-k)}$ and $x = (x_a, x^a) \in R^{\nu(N-1)} = R^n$. The corresponding canonically conjugate momentum operator is

$$\begin{aligned} p &= (p_a, p^a), \quad p_a = (p_1, \cdots, p_{k-1}), \quad p^a = (p^{(C_1)}, \cdots, p^{(C_k)}),\\ p_i &= \frac{\hbar}{i}\frac{\partial}{\partial x_i}, \quad p^{(C_\ell)} = (p_1^{(C_\ell)}, \cdots, p_{|C_\ell|-1}^{(C_\ell)}), \quad p_i^{(C_\ell)} = \frac{\hbar}{i}\frac{\partial}{\partial x_i^{(C_\ell)}}. \end{aligned}$$

Accordingly $\mathcal{H} = L^2(R^n)$ is decomposed:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_a \otimes \mathcal{H}^a, \quad \mathcal{H}_a = L^2(R_{x_a}^{\nu(k-1)}),\\ \mathcal{H}^a &= L^2(R_{x^a}^{\nu(N-k)}) = \mathcal{H}^{(C_1)} \otimes \cdots \otimes \mathcal{H}^{(C_k)}, \quad \mathcal{H}^{(C_\ell)} = L^2(R_{x^{(C_\ell)}}^{\nu(|C_\ell|-1)}). \end{aligned} \tag{7.11}$$

In this coordinates system, H_0 in (7.7) is decomposed:

$$\begin{aligned} H_0 &= T_a + H_0^a,\\ T_a &= -\sum_{\ell=1}^{k-1} \frac{\hbar^2}{2M_\ell}\Delta_{x_\ell},\\ H_0^a &= \sum_{\ell=1}^{k} H_0^{(C_\ell)}, \quad H_0^{(C_\ell)} = -\sum_{i=1}^{|C_\ell|-1} \frac{\hbar^2}{2\mu_i^{(C_\ell)}}\Delta_{x_i^{(C_\ell)}}, \end{aligned} \tag{7.12}$$

where Δ_{x_ℓ} and $\Delta_{x_i^{(C_\ell)}}$ are ν-dimensional Laplacians and M_ℓ and $\mu_i^{(C_\ell)}$ are the reduced masses. We introduce the inner product in the space $R^n = R^{\nu(N-1)}$ as in (7.9):

$$\begin{aligned}\langle x, y\rangle &= \langle (x_a, x^a), (y_a, y^a)\rangle = \langle x_a, y_a\rangle + \langle x^a, y^a\rangle \\ &= \sum_{\ell=1}^{k-1} M_\ell x_\ell \cdot y_\ell + \sum_{\ell=1}^{k} \sum_{i=1}^{|C_\ell|-1} \mu_i^{(C_\ell)} x_i^{(C_\ell)} \cdot y_i^{(C_\ell)},\end{aligned} \tag{7.13}$$

and velocity operator

$$v = (v_a, v^a) = M^{-1}p = (m_a^{-1}p_a, (\mu^a)^{-1}p^a), \tag{7.14}$$

where

$$M = \begin{pmatrix} m_a & 0 \\ 0 & \mu^a \end{pmatrix} \tag{7.15}$$

is the $n = \nu(N-1)$ dimensional diagonal mass matrix whose diagonals are given by $M_1, \cdots, M_{k-1}, \mu_1^{(C_1)}, \cdots, \mu_{|C_k|-1}^{(C_k)}$. Then H_0 is written as

$$H_0 = \frac{1}{2}\langle v, v\rangle = T_a + H_0^a = \frac{1}{2}\langle v_a, v_a\rangle + \frac{1}{2}\langle v^a, v^a\rangle. \tag{7.16}$$

We need a notion of order in the set of cluster decompositions. A cluster decomposition b is called a refinement of a cluster decomposition a, iff any $C_\ell \in b$ is a subset of some $D_k \in a$. When b is a refinement of a we denote this as $b \le a$. $b \not\le a$ is its negation: some cluster $C_\ell \in b$ is not a subset of any $D_k \in a$. Thus for a pair $\alpha = \{i, j\}$, $\alpha \le a$ means that $\alpha = \{i, j\} \subset D_k$ for some $D_k \in a$, and $\alpha \not\le a$ means that $\alpha = \{i, j\} \not\subset D_k$ for any $D_k \in a$. $b < a$ means that $b \le a$ but $b \ne a$.

We decompose the potential term V in (7.7) as

$$\sum_\alpha V_\alpha(x_\alpha) = V_a + I_a, \tag{7.17}$$

where

$$\begin{aligned} V_a &= \sum_{C_\ell \in a} V_{C_\ell}, \\ V_{C_\ell} &= \sum_{\alpha \subset C_\ell} V_\alpha(x_\alpha), \\ I_a &= \sum_{\alpha \not\le a} V_\alpha(x_\alpha). \end{aligned} \tag{7.18}$$

By definition, V_{C_ℓ} depends only on the variable $x^{(C_\ell)}$ inside the cluster C_ℓ. Similarly, V_a depends only on the variable $x^a = (x^{(C_1)}, \cdots, x^{(C_k)}) \in R^{\nu(N-|a|)}$, while I_a depends on all components of the variable x.

Then H in (7.7) is decomposed:

$$\begin{aligned} H &= H_a + I_a = T_a \otimes I + I \otimes H^a + I_a, \\ H_a &= H - I_a = T_a \otimes I + I \otimes H^a, \\ H^a &= H_0^a + V_a = \sum_{C_\ell \in a} H^{(C_\ell)}, \quad H^{(C_\ell)} = H_0^{(C_\ell)} + V_{C_\ell}, \end{aligned} \tag{7.19}$$

where T_a is an operator in $\mathcal{H}_a = L^2(R^{\nu(k-1)}_{x_a})$, H^a and H^a_0 are operators in $\mathcal{H}^a = L^2(R^{\nu(N-k)}_{x^a})$, and $H^{(C_\ell)}$ and $H^{(C_\ell)}_0$ are operators in $\mathcal{H}^{(C_\ell)} = L^2(R^{\nu(|C_\ell|-1)}_{x^{(C_\ell)}})$.

We denote by P_a the orthogonal projection onto the pure point spectral subspace (or eigenspace) $\mathcal{H}^a_{pp} = \mathcal{H}_{pp}(H^a)(\subset \mathcal{H}^a)$ for H^a. We use the same notation P_a for the obvious extention $I \otimes P_a$ to the total space $\mathcal{H}$. For $|a| = N$, we set $P_a = I$. Let $M = 1, 2, \cdots$ and P^M_a denote an M-dimensional partial projection of P_a such that $\text{s-lim}_{M\to\infty} P^M_a = P_a$. We define for $\ell = 1, \cdots, N-1$ and an ℓ-dimensional multi-index $M = (M_1, \cdots, M_\ell)$ $(M_j \geq 1)$

$$\widehat{P}^M_\ell = \left(I - \sum_{|a_\ell|=\ell} P^{M_\ell}_{a_\ell}\right) \cdots \left(I - \sum_{|a_2|=2} P^{M_2}_{a_2}\right)(I - P^{M_1}). \tag{7.20}$$

(Note that for $|a| = 1$, $a = \{C\}$ with $C = \{1, 2, \cdots, N\}$. Thus P^{M_1} is an M_1-dimensional partial projection into the eigenspace of H.) We further define for a $|a|$-dimensional multi-index $M_a = (M_1, \cdots, M_{|a|-1}, M_{|a|}) = (\widehat{M_a}, M_{|a|})$

$$\widetilde{P}^{M_a}_a = P^{M_{|a|}}_a \widehat{P}^{\widehat{M_a}}_{|a|-1}, \quad 2 \leq |a| \leq N. \tag{7.21}$$

Then it is clear that

$$\sum_{2\leq|a|\leq N} \widetilde{P}^{M_a}_a = \widehat{P}^{M_1}_1 = I - P^{M_1}, \tag{7.22}$$

provided that the component M_j of M_a depends only on the number j but not on a. In the following we use such M_a's only.

Related with those notions, we denote by $\mathcal{H}_c = \mathcal{H}_c(H)$ the orthogonal complement $\mathcal{H}_{pp}(H)^\perp$ of the eigenspace $\mathcal{H}_{pp} = \mathcal{H}_{pp}(H)$ for the total Hamiltonian H. Namely $\mathcal{H}_c(H)$ is the continuous spectral subspace for H. We note that $\mathcal{H}_c(H) = (I - P_a)\mathcal{H}$ for a unique a with $|a| = 1$, and that for $f \in \mathcal{H}$, $(I - P^{M_1})f \to (I - P_a)f \in \mathcal{H}_c(H)$ as $M_1 \to \infty$. We use freely the notations of functional analysis for selfadjoint operators, e.g. $E_H(\Delta)$ is the spectral measure for H.

To state a theorem due to Enss [10], we introduce an assumption:

Assumption 7.2 *For any cluster decomposition a with $2 \leq |a| \leq N-1$ and any integer $M = 1, 2, \cdots$,*

$$\||x^a|^2 P^M_a\| < \infty. \tag{7.23}$$

This assumption is concerned with the decay rate of eigenvectors of subsystem Hamiltonians. Since it is known that non-threshold eigenvectors decay exponentially (see Froese and Herbst [13]), this assumption is the one about threshold eigenvectors.

Let v_a, as above, denote the velocity operator between the clusters in a. It is expressed as $v_a = m_a^{-1} p_a$ for some $\nu(|a|-1)$-dimensional diagonal mass matrix m_a. Then we can state the theorem, which is the same as Theorem 3.2.

Theorem 7.3 *([10]) Let $N \geq 2$ and let H be the Hamiltonian H in (7.7) or (7.19) for an N-body quantum-mechanical system. Let Assumptions 7.1 and 7.2 be satisfied. Let $f \in \mathcal{H}$. Then there exist a sequence $t_m \to \pm\infty$ (as $m \to \pm\infty$) and a sequence M_a^m of multi-indices whose components all tend to ∞ as $m \to \pm\infty$ such that for all cluster decompositions a with $2 \leq |a| \leq N$, for all $\varphi \in C_0^\infty(R_{x_a}^{\nu(|a|-1)})$, $R > 0$, and $\alpha = \{i,j\} \not\leq a$*

$$\left\| \frac{|x^a|^2}{t_m^2} \widetilde{P}_a^{M_a^m} e^{-it_m H/\hbar} f \right\| \to 0 \tag{7.24}$$

$$\|F(|x_\alpha| < R)\widetilde{P}_a^{M_a^m} e^{-it_m H/\hbar} f\| \to 0 \tag{7.25}$$

$$\|(\varphi(x_a/t_m) - \varphi(v_a))\widetilde{P}_a^{M_a^m} e^{-it_m H/\hbar} f\| \to 0 \tag{7.26}$$

as $m \to \pm\infty$. Here $F(S)$ is the characteristic function of the set defined by the condition S.

We denote the sum of the sets of thresholds and eigenvalues of H by $\mathcal{T}$:

$$\mathcal{T} = \bigcup_{1\leq|a|\leq N} \sigma_p(H^a) = \tilde{\mathcal{T}} \cup \sigma_p(H), \quad \tilde{\mathcal{T}} = \bigcup_{2\leq|a|\leq N} \sigma_p(H^a) \tag{7.27}$$

where

$$\sigma_p(H^a) = \{\tau_1 + \cdots + \tau_{|a|} \mid \tau_\ell \in \sigma_p(H^{(C_\ell)}) \ (C_\ell \in a)\} \tag{7.28}$$

is the set of eigenvalues of a subsystem Hamiltonian $H^a = \sum_{C_\ell \in a} H^{(C_\ell)}$. For $|a| = N$ we define $\sigma_p(H^a) = \{0\}$. Similarly $\mathcal{T}_a$ and $\tilde{\mathcal{T}}_a$ are defined:

$$\mathcal{T}_a = \bigcup_{b\leq a} \sigma_p(H^b) = \tilde{\mathcal{T}}_a \cup \sigma_p(H^a), \quad \tilde{\mathcal{T}}_a = \bigcup_{b<a} \sigma_p(H^b). \tag{7.29}$$

It is known (Froese and Herbst [13]) that these sets are subsets of $(-\infty, 0]$. Further these sets form bounded, closed and countable subsets of R^1, and $\sigma_p(H^a)$ accumulates only at $\tilde{\mathcal{T}}_a$ (see Cycon *et al.* [5]).

We use the notation $\Delta \subset\subset \Delta'$ for Borel sets $\Delta, \Delta' \subset R^k$ to mean that the closure $\bar{\Delta}$ of Δ is compact in R^k and is a subset of the interior of Δ'.

7.2 Scattering spaces

In the following we consider the case $t \to \infty$ only. The other case $t \to -\infty$ is treated similarly. We also choose a unit system such that $\hbar = 1$. We use the notation $f(t) \sim g(t)$ as $t \to \infty$ to mean that $\|f(t) - g(t)\| \to 0$ as $t \to \infty$ for $\mathcal{H}$-valued functions $f(t)$ and $g(t)$ of $t > 1$.

Definition 7.4 *Let real numbers r, σ, δ and a cluster decomposition b satisfy $0 \le r \le 1$, $\sigma, \delta > 0$ and $2 \le |b| \le N$.*
i) Let $\Delta \subset\subset R^1 - \mathcal{T}$ be a closed set. We define $S_b^{r\sigma\delta}(\Delta)$ for $0 < r \le 1$ by

$$S_b^{r\sigma\delta}(\Delta) = \{f \in E_H(\Delta)\mathcal{H} \mid e^{-itH}f \sim \prod_{\alpha \not\le b} F(|x_\alpha| \ge \sigma t)F(|x^b| \le \delta t^r)e^{-itH}f \text{ as } t \to \infty\} \tag{7.30}$$

For $r = 0$ we define $S_b^{0\sigma}(\Delta)$ by

$$\begin{aligned} &S_b^{0\sigma}(\Delta) = \{f \in E_H(\Delta)\mathcal{H} \mid \\ &\lim_{R\to\infty} \limsup_{t\to\infty} \left\| e^{-itH}f - \prod_{\alpha \not\le b} F(|x_\alpha| \ge \sigma t)F(|x^b| \le R)e^{-itH}f \right\| = 0\}. \end{aligned} \tag{7.31}$$

We then define the localized scattering space $S_b^r(\Delta)$ of order $r \in (0, 1]$ for H as the closure of

$$\begin{aligned} &\bigcup_{\sigma>0} \bigcap_{\delta>0} S_b^{r\sigma\delta}(\Delta) = \{f \in E_H(\Delta)\mathcal{H} \mid \exists \sigma > 0, \forall \delta > 0 : \\ &e^{-itH}f \sim \prod_{\alpha \not\le b} F(|x_\alpha| \ge \sigma t)F(|x^b| \le \delta t^r)e^{-itH}f \text{ as } t \to \infty\}. \end{aligned} \tag{7.32}$$

$S_b^0(\Delta)$ is defined as the closure of

$$\begin{aligned} &\bigcup_{\sigma>0} S_b^{0\sigma}(\Delta) = \{f \in E_H(\Delta)\mathcal{H} \mid \exists \sigma > 0 : \\ &\lim_{R\to\infty} \limsup_{t\to\infty} \left\| e^{-itH}f - \prod_{\alpha \not\le b} F(|x_\alpha| \ge \sigma t)F(|x^b| \le R)e^{-itH}f \right\| = 0\}. \end{aligned} \tag{7.33}$$

ii) We define the scattering space S_b^r of order $r \in [0, 1]$ for H as the closure of

$$\bigcup_{\Delta \subset\subset R^1 - \mathcal{T}} S_b^r(\Delta). \tag{7.34}$$

We note that $S_b^{r\sigma\delta}(\Delta)$, $S_b^{0\sigma}(\Delta)$, $S_b^r(\Delta)$ and S_b^r define closed subspaces of $E_H(\Delta)\mathcal{H}$ and $\mathcal{H}_c(H)$, respectively.

Proposition 7.5 *Let $\Delta \subset\subset R^1 - \mathcal{T}$ and $f \in S_b^{r\sigma\delta}(\Delta)$ for $0 < r \le 1$ or $f \in S_b^{0\sigma}(\Delta)$ for $r = 0$ with $\sigma, \delta > 0$ and $2 \le |b| \le N$. Then the following limit relations hold:*

i) Let $\alpha \not\le b$. Then for $0 < r \le 1$ we have when $t \to \infty$

$$F(|x_\alpha| < \sigma t)F(|x^b| \le \delta t^r)e^{-itH}f \to 0. \tag{7.35}$$

For $r = 0$ we have

$$\lim_{R\to\infty} \limsup_{t\to\infty} \left\| F(|x_\alpha| < \sigma t)F(|x^b| \le R)e^{-itH}f \right\| = 0. \tag{7.36}$$

ii) For $0 < r \le 1$ we have when $t \to \infty$

$$F(|x^b| > \delta t^r)e^{-itH} f \to 0. \tag{7.37}$$

For $r = 0$

$$\lim_{R\to\infty} \limsup_{t\to\infty} \left\| F(|x^b| > R)e^{-itH} f \right\| = 0. \tag{7.38}$$

iii) There exists a sequence $t_m \to \infty$ as $m \to \infty$ depending on $f \in S_b^{r\sigma\delta}(\Delta)$ or $f \in S_b^{0\sigma}(\Delta)$ such that

$$\left\| \left(\varphi\left(x_b/t_m\right) - \varphi(v_b)\right) e^{-it_m H} f \right\| \to 0 \quad as \quad m \to \infty \tag{7.39}$$

for any function $\varphi \in C_0^\infty(R_{x_b}^{\nu(|b|-1)})$.

Proof: i) and ii) are clear from the definition of $S_b^{r\sigma\delta}(\Delta)$ or $S_b^{0\sigma}(\Delta)$. We prove iii). Since $f \in E_H(\Delta)\mathcal{H} \subset H_c(H)$, we have by (7.22), Theorem 3.2 and $f \in S_b^{r\sigma\delta}(\Delta)$ (or $f \in S_b^{0\sigma}(\Delta)$)

$$e^{-it_m H} f \sim \sum_{d\le b} \widetilde{P}_d^{M_d^m} e^{-it_m H} f \tag{7.40}$$

along some sequence $t_m \to \infty$ depending on f. On each state on the right-hand side (RHS) of (7.40), (7.26) with a replaced by d holds. By the restriction $d \le b$ in the sum of the RHS of (7.40), we obtain (7.39). □

The following propositions are obvious by definition.

Proposition 7.6 *Let $2 \le |b| \le N$. If $1 \ge r' \ge r > 0$, $\sigma \ge \sigma' > 0$ and $\delta' \ge \delta > 0$ and $\Delta \subset\subset R^1 - \mathcal{T}$, then $S_b^{0\sigma}(\Delta) \subset S_b^{0\sigma'}(\Delta)$, $S_b^{0\sigma}(\Delta) \subset S_b^{r\sigma\delta}(\Delta) \subset S_b^{r'\sigma'\delta'}(\Delta)$, $S_b^0(\Delta) \subset S_b^r(\Delta) \subset S_b^{r'}(\Delta)$, $S_b^0(\Delta) \subset S_b^r(\Delta) \subset S_b^r$, and $S_b^0 \subset S_b^r \subset S_b^{r'}$.*

Proposition 7.7 *Let b and b' be different cluster decompositions: $b \ne b'$. Then for any $0 \le r \le 1$, S_b^r and $S_{b'}^r$ are orthogonal mutually: $S_b^r \perp S_{b'}^r$.*

7.3 A partition of unity

To state a proposition that will play a fundamental role in our decomposition of continuous spectral subspace by S_b^1, we prepare some notations. Let b be a cluster decomposition with $2 \le |b| \le N$. For any two clusters C_1 and C_2 in b, we define a vector z_{b1} that connects the two centers of mass of the clusters C_1 and C_2. The number of such vectors when we move over all pairs of clusters in b is $k_b = \begin{pmatrix} |b| \\ 2 \end{pmatrix}$ in total. We denote these vectors by $z_{b1}, z_{b2}, \cdots, z_{bk_b}$.

Let z_{bk} $(1 \le k \le k_b)$ connect two clusters C_ℓ and C_m in b $(\ell \ne m)$. Then for any pair $\alpha = \{i, j\}$ with $i \in C_\ell$ and $j \in C_m$, the vector $x_\alpha = x_{ij}$ is expressed like $(z_{bk}, x^{(C_\ell)}, x^{(C_m)}) \in R^{\nu+\nu(|C_\ell|-1)+\nu(|C_m|-1)}$, where $x^{(C_\ell)}(\in R^{\nu(|C_\ell|-1)})$ and $x^{(C_m)}(\in R^{\nu(|C_m|-1)})$ are the positions of the particles i and j in C_ℓ and C_m, respectively. The vector expression

$x_\alpha = (z_{bk}, x^{(C_\ell)}, x^{(C_m)})$ is in the space $R^{\nu+\nu(|C_\ell|-1)+\nu(|C_m|-1)}$. If we express it in the larger space $R^{\nu}_{z_{bk}} \times R^{\nu(N-|b|)}_{x^b}$, it would be $x_\alpha = (z_{bk}, x^b)$, and $|x_\alpha|^2 = |z_{bk}|^2 + |x^b|^2$. Thus if $|z_{bk}|^2$ is sufficiently large compared to $|x^b|^2 \geq |x^{(C_\ell)}|^2 + |x^{(C_m)}|^2$, e.g. if $|z_{bk}|^2 > \rho > 0$ and $|x^b|^2 < \theta$ with $\rho \gg \theta > 0$ (which means that ρ/θ is sufficiently large), then $|x_\alpha|^2 > \rho/2$ for all $\alpha \not\leq b$.

Next if $c < b$ and $|c| = |b| + 1$, then just one cluster, say $C_\ell \in b$, is decomposed into two clusters C'_ℓ and C''_ℓ in c, and other clusters in b remain the same in the finer cluster decomposition c. In this case, we can choose just one vector z_{ck} ($1 \leq k \leq k_c$) that connects clusters C'_ℓ and C''_ℓ in c, and we can express $x^b = (z_{ck}, x^c)$. The norm of this vector is written as

$$|x^b|^2 = |z_{ck}|^2 + |x^c|^2. \tag{7.41}$$

Similarly the norm of $x = (x_b, x^b)$ is written as

$$|x|^2 = |x_b|^2 + |x^b|^2. \tag{7.42}$$

We recall that norm is defined, as usual, from the inner product defined by (7.13) which changes in accordance with the cluster decomposition used in each context. E.g., in (7.41), the left-hand side (LHS) is defined by using (7.13) for the cluster decomposition b and the RHS is by using (7.13) for c.

With these preparations, we state the following lemma, which is partially a repetition of [24], Lemma 2.1. We define subsets $T_b(\rho, \theta)$ and $\tilde{T}_b(\rho, \theta)$ of $R^n = R^{\nu(N-1)}$ for cluster decompositions b with $2 \leq |b| \leq N$ and real numbers ρ, θ with $1 > \rho, \theta > 0$:

$$T_b(\rho, \theta) = \left(\bigcap_{k=1}^{k_b} \{x \mid |z_{bk}|^2 > \rho |x|^2\} \right) \cap \{x \mid |x_b|^2 > (1-\theta)|x|^2\}, \tag{7.43}$$

$$\tilde{T}_b(\rho, \theta) = \left(\bigcap_{k=1}^{k_b} \{x \mid |z_{bk}|^2 > \rho\} \right) \cap \{x \mid |x_b|^2 > 1-\theta\}. \tag{7.44}$$

Subsets S and S_θ ($\theta > 0$) of $R^n = R^{\nu(N-1)}$ are defined by

$$\begin{aligned} S &= \{x \mid |x|^2 \geq 1\}, \\ S_\theta &= \{x \mid 1 + \theta \geq |x|^2 \geq 1\}. \end{aligned}$$

Lemma 7.8 *Suppose that constants* $1 \geq \theta_1 > \rho_j > \theta_j > \rho_N > 0$ *satisfy* $\theta_{j-1} \geq \theta_j + \rho_j$ *for* $j = 2, 3, \cdots, N-1$. *Then the followings hold:*

i)

$$S \subset \bigcup_{2 \leq |b| \leq N} T_b(\rho_{|b|}, \theta_{|b|}). \tag{7.45}$$

ii) Let $\gamma_j > 1$ $(j = 1, 2)$ *satisfy*

$$\gamma_1 \gamma_2 < r_0 := \min_{2 \leq j \leq N-1} \{\rho_j / \theta_j\}. \tag{7.46}$$

If $b \not\leq c$ with $|b| \geq |c|$, then

$$T_b(\gamma_1^{-1}\rho_{|b|}, \gamma_2\theta_{|b|}) \cap T_c(\gamma_1^{-1}\rho_{|c|}, \gamma_2\theta_{|c|}) = \emptyset. \tag{7.47}$$

iii) For $\gamma > 1$ and $2 \leq |b| \leq N$

$$\begin{aligned} T_b(\rho_{|b|}, \theta_{|b|}) \cap S_{\theta_{N-1}} &\subset \tilde{T}_b(\rho_{|b|}, \theta_{|b|}) \cap S_{\theta_{N-1}} \\ &\subset\subset \tilde{T}_b(\gamma^{-1}\rho_{|b|}, \gamma\theta_{|b|}) \cap S_{\theta_{N-1}} \\ &\subset T_b({\gamma'_1}^{-1}\rho_{|b|}, \gamma'_2\theta_{|b|}) \cap S_{\theta_{N-1}}, \end{aligned} \tag{7.48}$$

where

$$\gamma'_1 = \gamma(1+\theta_{N-1}), \quad \gamma'_2 = (1+\gamma)(1+\theta_{N-1})^{-1}. \tag{7.49}$$

iv) If $\frac{2\gamma'_1\gamma'_2}{2-\gamma'_1} < r_0$, then for $2 \leq |b| \leq N$

$$T_b({\gamma'_1}^{-1}\rho_{|b|}, \gamma'_2\theta_{|b|}) \subset \{x \mid |x_\alpha|^2 > \rho_{|b|}|x|^2/2 \text{ for all } \alpha \not\leq b\}. \tag{7.50}$$

v) If $\gamma(1+\gamma) < r_0$ and $b \not\leq c$ with $|b| \geq |c|$, then

$$T_b({\gamma'_1}^{-1}\rho_{|b|}, \gamma'_2\theta_{|b|}) \cap T_c({\gamma'_1}^{-1}\rho_{|c|}, \gamma'_2\theta_{|c|}) = \emptyset. \tag{7.51}$$

Proof: To prove (7.45), suppose that $|x|^2 \geq 1$ and x does not belong to the set

$$A = \bigcup_{2\leq|b|\leq N-1} \left[\left(\bigcap_{k=1}^{k_b} \{x \mid |z_{bk}|^2 > \rho_{|b|}|x|^2\} \right) \cap \{x \mid |x_b|^2 > (1-\theta_{|b|})|x|^2\} \right].$$

Under this assumption, we prove $|x_\alpha|^2 > \rho_N|x|^2$ for all pairs $\alpha = \{i,j\}$. (Note that z_{bk} for $|b| = N$ equals some x_α.) Let $|b| = 2$ and write $x = (z_{b1}, x^b)$. Then by (7.41), $1 \leq |x|^2 = |z_{b1}|^2 + |x^b|^2$. Since x belongs to the complement A^c of the set A, we have $|z_{b1}|^2 \leq \rho_{|b|}|x|^2$ or $|x_b|^2 \leq (1-\theta_{|b|})|x|^2$. If $|z_{b1}|^2 \leq \rho_{|b|}|x|^2$, then $|x^b|^2 = |x|^2 - |z_{b1}|^2 \geq (1-\rho_{|b|})|x|^2 \geq (\theta_1 - \rho_{|b|})|x|^2 \geq \theta_{|b|}|x|^2$ by $\theta_{j-1} \geq \theta_j + \rho_j$. Thus $|x_b|^2 = |x|^2 - |x^b|^2 \leq (1-\theta_{|b|})|x|^2$ for all b with $|b| = 2$.

Next let $|c| = 3$ and assume $|x_c|^2 > (1-\theta_{|c|})|x|^2$. Then by $x \in A^c$, we can choose z_{ck} with $1 \leq k \leq k_c$ such that $|z_{ck}|^2 \leq \rho_{|c|}|x|^2$. Let C_ℓ and C_m be two clusters in c connected by z_{ck}, and let b be the cluster decomposition obtained by combining C_ℓ and C_m into one cluster with retaining other clusters of c in b. Then $|b| = 2$, $x^b = (z_{ck}, x^c)$, and $|x^b|^2 = |z_{ck}|^2 + |x^c|^2$. Thus $|x_b|^2 = |x|^2 - |x^b|^2 = |x|^2 - |z_{ck}|^2 - |x^c|^2 = |x_c|^2 - |z_{ck}|^2 > (1-\theta_{|c|} - \rho_{|c|})|x|^2 \geq (1-\theta_{|b|})|x|^2$, which contradicts the result of the previous step. Thus $|x_c|^2 \leq (1-\theta_{|c|})|x|^2$ for all c with $|c| = 3$.

Repeating this procedure, we finally arrive at $|x_d|^2 \leq (1-\theta_{|d|})|x|^2$, thus $|x^d|^2 = |x|^2 - |x_d|^2 \geq \theta_{|d|}|x|^2 > \rho_N|x|^2$ for all d with $|d| = N-1$. Namely $|x_\alpha|^2 > \rho_N|x|^2$ for all pairs $\alpha = \{i,j\}$. The proof of (7.45) is complete.

We next prove (7.47). By $b \not\leq c$, we can take a pair $\alpha = \{i, j\}$ and clusters $C_\ell, C_m \in c$ such that $\alpha \leq b$, $i \in C_\ell$, $j \in C_m$, and $\ell \neq m$. Then we can write $x_\alpha = (z_{ck}, x^c)$ for some $1 \leq k \leq k_c$. Thus if there is $x \in T_b(\gamma_1^{-1}\rho_{|b|}, \gamma_2\theta_{|b|}) \cap T_c(\gamma_1^{-1}\rho_{|c|}, \gamma_2\theta_{|c|})$, then

$$\gamma_2\theta_{|b|}|x|^2 > |x^b|^2 \geq |x_\alpha|^2 = |z_{ck}|^2 + |x^c|^2 \geq |z_{ck}|^2 > \gamma_1^{-1}\rho_{|c|}|x|^2. \tag{7.52}$$

But since $|b| \geq |c|$, we have $\rho_{|c|} > \gamma_1\gamma_2\theta_{|b|}$ when $|b| = |c|$ by (7.46), and $\rho_{|c|} > \gamma_1\gamma_2\theta_{|c|} \geq \gamma_1\gamma_2(\theta_{|b|} + \rho_{|b|}) > \gamma_1\gamma_2\theta_{|b|}$ when $|c| < |b|$ by $\theta_{j-1} \geq \theta_j + \rho_j$, which both contradict the inequality (7.52). This completes the proof of (7.47).

(7.48) follows by a simple calculation from the inequality $|x|^2(1 + \theta_{N-1})^{-1} \leq 1$ that holds on $S_{\theta_{N-1}}$. (7.50) follows from the relation $|x_\alpha|^2 = |z_{bk}|^2 + |x^b|^2$ stated before the lemma, and (7.51) from $\gamma_1'\gamma_2' = \gamma(1+\gamma)$ and ii). □

In the followings we fix constants $\gamma > 1$ and $1 \geq \theta_1 > \rho_j > \theta_j > \rho_N > 0$ such that

$$\theta_{j-1} \geq \theta_j + \rho_j \quad (j = 2, 3, \cdots, N-1), \tag{7.53}$$

$$\max\left\{\gamma(1+\gamma), \frac{2\gamma_1'\gamma_2'}{2 - \gamma_1'}\right\} < r_0 = \min_{2 \leq j \leq N-1}\{\rho_j/\theta_j\}, \tag{7.54}$$

where γ_j' $(j = 1, 2)$ are defined by (7.49).

Let $\rho(\lambda) \in C^\infty(R^1)$ be such that $0 \leq \rho(\lambda) \leq 1$, $\rho(\lambda) = 1$ $(\lambda \leq -1)$, $\rho(\lambda) = 0$ $(\lambda \geq 0)$, and $\rho'(\lambda) \leq 0$. Then we define functions $\phi_\sigma(\lambda < \tau)$ and $\phi_\sigma(\lambda > \tau)$ of $\lambda \in R^1$ by

$$\phi_\sigma(\lambda < \tau) = \rho((\lambda - (\tau + \sigma))/\sigma), \tag{7.55}$$

$$\phi_\sigma(\lambda > \tau) = 1 - \phi_\sigma(\lambda < \tau - \sigma) \tag{7.56}$$

for constants $\sigma > 0, \tau \in R^1$. We note that $\phi_\sigma(\lambda < \tau)$ and $\phi_\sigma(\lambda > \tau)$ satisfy

$$\phi_\sigma(\lambda < \tau) = \begin{cases} 1 & (\lambda \leq \tau) \\ 0 & (\lambda \geq \tau + \sigma) \end{cases} \tag{7.57}$$

$$\phi_\sigma(\lambda > \tau) = \begin{cases} 0 & (\lambda \leq \tau - \sigma) \\ 1 & (\lambda \geq \tau) \end{cases} \tag{7.58}$$

$$\begin{aligned} &\phi_\sigma'(\lambda < \tau) = \frac{d}{d\lambda}\phi_\sigma(\lambda < \tau) \leq 0, \\ &\phi_\sigma'(\lambda > \tau) \geq 0. \end{aligned} \tag{7.59}$$

We define for a cluster decomposition b with $2 \leq |b| \leq N$

$$\varphi_b(x_b) = \prod_{k=1}^{k_b} \phi_\sigma(|z_{bk}|^2 > \rho_{|b|})\phi_\sigma(|x_b|^2 > 1 - \theta_{|b|}), \tag{7.60}$$

where $\sigma > 0$ is fixed as

$$0 < \sigma < \min_{2 \leq j \leq N-1}\{(1 - \gamma^{-1})\rho_N, (1 - \gamma^{-1})\rho_j, (\gamma - 1)\theta_j\}. \tag{7.61}$$

Then $\varphi_b(x_b)$ satisfies for $x \in S_{\theta_{N-1}}$

$$\varphi_b(x_b) = \begin{cases} 1 & \text{for} \quad x \in \tilde{T}_b(\rho_{|b|}, \theta_{|b|}), \\ 0 & \text{for} \quad x \notin \tilde{T}_b(\gamma^{-1}\rho_{|b|}, \gamma\theta_{|b|}). \end{cases} \tag{7.62}$$

We set for $|b| = k$ $(k = 2, 3, \cdots, N)$

$$J_b(x) = \varphi_b(x_b)\left(1 - \sum_{|b_{k-1}|=k-1} \varphi_{b_{k-1}}(x_{b_{k-1}})\right)\cdots\left(1 - \sum_{|b_2|=2}\varphi_{b_2}(x_{b_2})\right). \tag{7.63}$$

By v) and iii) of Lemma 7.8 and (7.62), the sums on the RHS remain only in the case $b < b_j$ for $j = k-1, \cdots, 2$ and $x \in S_{\theta_{N-1}}$:

$$J_b(x) = \varphi_b(x_b)\left(1 - \sum_{|b_{k-1}|=k-1, b<b_{k-1}} \varphi_{b_{k-1}}(x_{b_{k-1}})\right)\cdots\left(1 - \sum_{|b_2|=2, b<b_2}\varphi_{b_2}(x_{b_2})\right). \tag{7.64}$$

Thus $J_b(x)$ is a function of the variable x_b only:

$$J_b(x) = J_b(x_b) \quad \text{when} \quad x = (x_b, x^b) \in S_{\theta_{N-1}}. \tag{7.65}$$

We also note that the supports of φ_{b_j} in each sum on the RHS of (7.63) are disjoint mutually in $S_{\theta_{N-1}}$ by iii) and v) of Lemma 7.8. By (7.45) and (7.48) of lemma 7.8, and the definition (7.60)-(7.63) of $J_b(x_b)$, we therefore have

$$\sum_{2\le|b|\le N} J_b(x_b) = 1 \quad \text{on} \quad S_{\theta_{N-1}}.$$

We have constructed a partition of unity on $S_{\theta_{N-1}}$:

Proposition 7.9 *Let real numbers $1 \ge \theta_1 > \rho_j > \theta_j > \rho_N > 0$ satisfy $\theta_{j-1} \ge \theta_j + \rho_j$ for $j = 2, 3, \cdots, N-1$. Assume that (7.54) holds and let $J_b(x_b)$ be defined by (7.60)-(7.64). Then we have*

$$\sum_{2\le|b|\le N} J_b(x_b) = 1 \quad on \quad S_{\theta_{N-1}}. \tag{7.66}$$

$J_b(x_b)$ is a C^∞ function of x_b and satisfies $0 \le J_b(x_b) \le 1$. Further on $\operatorname{supp} J_b \cap S_{\theta_{N-1}}$ *we have*

$$|x_\alpha|^2 > \rho_{|b|}|x|^2/2 \tag{7.67}$$

for any pair $\alpha \not\le b$, and

$$\sup_{x\in R^n, 2\le|b|\le N} |\nabla_{x_b} J_b(x_b)| < \infty \tag{7.68}$$

for each fixed $\sigma > 0$ in (7.60)-(7.61).

Proof: We have only to see (7.67) and (7.68). But (7.67) is clear by (7.48), (7.50), (7.54), (7.62) and (7.63), and (7.68) follows from (7.56), (7.60) and (7.64). □

7.4 A decomposition of continuous spectral subspace

The following theorem gives a decomposition of $\mathcal{H}_c(H)$ by scattering spaces S_b^1 $(2 \le |b| \le N)$.

Theorem 7.10 *Let Assumptions 7.1 and 7.2 be satisfied. Then we have*

$$\mathcal{H}_c(H) = \bigoplus_{2\le|b|\le N} S_b^1. \tag{7.69}$$

Proof: Since the set

$$\bigcup_{\Delta\subset\subset R^1-\mathcal{T}} E_H(\Delta)\mathcal{H}$$

is dense in $\mathcal{H}_c(H)$, and S_b^1 $(2 \le |b| \le N)$ are closed and mutually orthogonal, it suffices to prove that any $\Phi(H)f$ with $\Phi \in C_0^\infty(R^1 - \mathcal{T})$ and $f \in \mathcal{H}$ can be decomposed as a sum of the elements f_b^1 in S_b^1: $\Phi(H)f = \sum_{2\le|b|\le N} f_b^1$.

We divide the proof into two steps. In the first step I), we prove existence of certain time limits. In the second step II), we prove existence of some “boundary values” of those limits, and conclude the proof of decomposition (7.69).

I) Existence of some time limits:

We decompose $\Phi(H)f$ as a finite sum: $\Phi(H)f = \sum_{j_0}^{\text{finite}} \psi_{j_0}(H)f$, where $\psi_{j_0} \in C_0^\infty(R^1 - \mathcal{T})$. In the step I), we will prove the existence of the limit

$$\lim_{t\to\infty} \sum_{\ell=1}^{L} e^{itH} G_{b,\lambda_\ell}(t)^* J_b(v_b/r_\ell) G_{b,\lambda_\ell}(t) e^{-itH} \psi_{j_0}(H)f, \tag{7.70}$$

under the assumption that $\operatorname{supp}\psi_{j_0} \subset \tilde{\Delta} \subset\subset \Delta$ for some intervals $\tilde{\Delta} \subset\subset \Delta \subset\subset R^1 - \mathcal{T}$ with $E \in \Delta$ and $\operatorname{diam}\Delta < d(E)$, where $d(E) > 0$ is some small constant depending on $E \in R^1 - \mathcal{T}$ and $\operatorname{diam} S$ denotes the diameter of a set $S \subset R^1$. The relevant factors in (7.70) will be defined in the course of the proof. We will write f for $\psi_{j_0}(H)f$ in the followings.

We take $\psi \in C_0^\infty(R^1)$ such that $\psi(\lambda) = 1$ for $\lambda \in \tilde{\Delta}$ and $\operatorname{supp}\psi \subset \Delta$ for the intervals $\tilde{\Delta} \subset\subset \Delta$ above. Then $f = \psi(H)f = E_H(\Delta)f \in E_H(\Delta)\mathcal{H} \subset \mathcal{H}_c(H)$ and $e^{-itH}f = \psi(H)e^{-itH}f$. Thus we can use the decomposition (7.22) for the sequences t_m and M_b^m in Theorem 7.3:

$$e^{-it_mH}f = \psi(H)e^{-it_mH}f = \psi(H)\sum_{2\le|d|\le N} \widetilde{P}_d^{M_d^m} e^{-it_mH}f. \tag{7.71}$$

By Theorem 7.3-(7.25)

$$\psi(H)\widetilde{P}_d^{M_d^m} e^{-it_mH}f \sim \psi(H_d)\widetilde{P}_d^{M_d^m} e^{-it_mH}f \tag{7.72}$$

as $m \to \infty$. Since

$$\widetilde{P}_d^{M_d^m} = P_d^{M_{|d|}^m}\widehat{P}_{|d|-1}^{\widehat{M}_d^m}, \quad P_d^{M_{|d|}^m} = \sum_{j=1}^{M_{|d|}^m} P_{d,E_j}, \tag{7.73}$$

where P_{d,E_j} is one dimensional eigenprojection for H^d with eigenvalue E_j, the RHS of (7.72) equals

$$\sum_{j=1}^{M^m_{|d|}} \psi(T_d + E_j) P_{d,E_j} \widehat{P}^{\widehat{M^m_d}}_{|d|-1} e^{-it_m H} f. \tag{7.74}$$

By $\operatorname{supp}\psi \subset \Delta \subset\subset R^1 - \mathcal{T}$, $E_j \in \mathcal{T}$, and $T_d \geq 0$, we can take constants $\Lambda_d > \lambda_d > 0$ independent of $j = 1, 2, \cdots$ such that $\Lambda_d \geq T_d \geq \lambda_d$ if $\psi(T_d + E_j) \neq 0$. Set $\Lambda_0 = \max_d \Lambda_d > \lambda_0 = \min_d \lambda_d > 0$ and

$$\Sigma(E) = \{E - \lambda \mid \lambda \in \mathcal{T}, E \geq \lambda\}. \tag{7.75}$$

Note that we can take $\Lambda_0 > \lambda_0 > 0$ so that

$$\Sigma(E) \subset\subset (\lambda_0, \Lambda_0) \subset (0, \infty). \tag{7.76}$$

Let $\Psi \in C_0^\infty(R^1)$ satisfy $\Psi(\lambda) = 1$ for $\lambda \in [\lambda_0, \Lambda_0]$ and $\operatorname{supp}\Psi \subset [\lambda_0 - \kappa, \Lambda_0 + \kappa]$ for some small constant $\kappa > 0$ such that the set $[\lambda_0', \lambda_0] \cup [\Lambda_0, \Lambda_0']$ is bounded away from $\Sigma(E)$, where $\lambda_0' = \lambda_0 - 2\kappa > 0$ and $\Lambda_0' = \Lambda_0 + 2\kappa$. Then the RHS of (7.74) equals for any $m = 1, 2, \cdots$

$$\Psi^2(T_d)\psi(H_d)\widetilde{P}_d^{M_d^m} e^{-it_m H} f. \tag{7.77}$$

On the other hand, by Theorem 7.3-(7.24) and (7.26), we have

$$\frac{|x^d|^2}{t_m^2}\psi(H_d)\widetilde{P}_d^{M_d^m} e^{-it_m H} f \sim 0 \tag{7.78}$$

and

$$\Psi^2(T_d)\psi(H_d)\widetilde{P}_d^{M_d^m} e^{-it_m H} f \sim \Psi^2(|x_d|^2/(2t_m^2))\psi(H_d)\widetilde{P}_d^{M_d^m} e^{-it_m H} f \tag{7.79}$$

as $m \to \infty$, where to see (7.78) we used (7.24) and $i[H^d, |x^d|^2/t^2] = i[H_0^d, |x^d|^2/t^2] = 2A^d/t^2$ where $A^d = (x^d \cdot p^d + p^d \cdot x^d)/2$, and to see (7.79) the fact that $|x_d|^2/t_m^2$ and H_d commute asymptotically as $m \to \infty$ by (7.26). Thus by $|x|^2 = |x_d|^2 + |x^d|^2$ we have

$$\Psi^2(T_d)\psi(H_d)\widetilde{P}_d^{M_d^m} e^{-it_m H} f \sim \Psi^2(|x|^2/(2t_m^2))\psi(H_d)\widetilde{P}_d^{M_d^m} e^{-it_m H} f \tag{7.80}$$

as $m \to \infty$. From (7.71)-(7.72), (7.74), (7.77) and (7.80), we obtain

$$e^{-it_m H} f \sim \Psi^2(|x|^2/(2t_m^2))e^{-it_m H} f \tag{7.81}$$

as $m \to \infty$.

Let constants $\gamma > 1$ and $1 \geq \theta_1 > \rho_j > \theta_j > \rho_N > 0$ be fixed such that

$$\theta_{j-1} \geq \theta_j + \rho_j \quad (j = 2, \cdots, N-1), \tag{7.82}$$

$$\max\left\{\gamma(1+\gamma), \frac{2\gamma_1'\gamma_2'}{2-\gamma_1'}\right\} < r_0 = \min_{2\leq j\leq N-1}\{\rho_j/\theta_j\} \tag{7.83}$$

for γ_j' $(j = 1, 2)$ defined by (7.49). Set

$$\lambda_0'' = \lambda_0' \theta_{N-1} > 0 \tag{7.84}$$

with $\lambda_0' = \lambda_0 - 2\kappa$ defined above. Let $\tau_0 > 0$ satisfy

$$0 < 16\tau_0 < \lambda_0''(< \lambda_0' < \lambda_0). \tag{7.85}$$

We take a finite subset $\{\tilde{\lambda}_\ell\}_{\ell=1}^L$ of $\mathcal{T}$ such that

$$\mathcal{T} \subset \bigcup_{\ell=1}^{L} (\tilde{\lambda}_\ell - \tau_0, \tilde{\lambda}_\ell + \tau_0). \tag{7.86}$$

Then we can choose real numbers $\lambda_\ell \in R^1$, $\tau_\ell > 0$ $(\ell = 1, 2, \cdots, L)$ and $\sigma_0 > 0$ such that

$$\begin{aligned} &\tau_\ell < \tau_0, \quad \sigma_0 < \tau_0, \quad |\lambda_\ell - \tilde{\lambda}_\ell| < \tau_0, \\ &\mathcal{T} \subset \bigcup_{\ell=1}^{L} (\lambda_\ell - \tau_\ell, \lambda_\ell + \tau_\ell), \quad (\lambda_\ell - \tau_\ell, \lambda_\ell + \tau_\ell) \subset (\tilde{\lambda}_\ell - \tau_0, \tilde{\lambda}_\ell + \tau_0), \\ &\operatorname{dist}\{(\lambda_\ell - \tau_\ell, \lambda_\ell + \tau_\ell), (\lambda_k - \tau_k, \lambda_k + \tau_k)\} > 4\sigma_0 (> 0) \quad \text{for any } \ell \neq k. \end{aligned} \tag{7.87}$$

We note that for $\ell = 1, \cdots, L$

$$\{\Lambda \mid \tau_\ell \leq |\Lambda - (E - \lambda_\ell)| \leq \tau_\ell + 4\sigma_0\} \cap \Sigma(E) = \emptyset. \tag{7.88}$$

Now let the intervals Δ and $\tilde{\Delta}$ be so small that

$$\operatorname{diam} \tilde{\Delta} < \operatorname{diam} \Delta < \tilde{\tau}_0 := \min_{1 \leq \ell \leq L} \{\sigma_0, \tau_\ell\}. \tag{7.89}$$

Returning to (7.74), we have

$$T_d + E_j \in \operatorname{supp} \psi, \tag{7.90}$$

if $\psi(T_d + E_j) \neq 0$ in (7.74). By $\operatorname{supp} \psi \subset \Delta$, $\operatorname{diam} \Delta < \tilde{\tau}_0$, and $E \in \Delta$, we have from (7.90)

$$-\tilde{\tau}_0 \leq T_d - (E - E_j) \leq \tilde{\tau}_0. \tag{7.91}$$

Thus we have asymptotically on each state in (7.74)

$$-2\tilde{\tau}_0 \leq \frac{|x|^2}{t_m^2} - 2(E - E_j) \leq 2\tilde{\tau}_0. \tag{7.92}$$

By (7.87), $E_j \in \mathcal{T}$ is included in just one set $(\lambda_\ell - \tau_\ell, \lambda_\ell + \tau_\ell)$ for some $\ell = \ell(j)$ with $1 \leq \ell(j) \leq L$. Since $|E_j - \lambda_{\ell(j)}| < \tau_{\ell(j)}$, we have using (7.89)

$$-2\tau_{\ell(j)} - 2\sigma_0 \leq \frac{|x|^2}{t_m^2} - 2(E - \lambda_{\ell(j)}) \leq 2\tau_{\ell(j)} + 2\sigma_0 \tag{7.93}$$

on each state in (7.74). Thus

$$\sum_{\ell=1}^{L} \phi_{\sigma_0}^2(\big||x|^2/t_m^2 - 2(E-\lambda_\ell)\big| < 2\tau_\ell + 2\sigma_0) = 1$$

asymptotically as $m \to \infty$ on (7.74). Now by the same reasoning that led us to (7.81), we see that (7.71) asymptotically equals as $m \to \infty$

$$\sum_{\ell=1}^{L} \sum_{2\le |d| \le N} \phi_{\sigma_0}^2(\big||x|^2/t_m^2 - 2(E-\lambda_\ell)\big| < 2\tau_\ell + 2\sigma_0)\Psi^2(|x|^2/(2t_m^2))\widetilde{P}_d^{M_d^m} e^{-it_m H} f. \quad (7.94)$$

Since $\phi_{\sigma_0}(||x|^2/t_m^2 - 2(E-\lambda_\ell)| < 2\tau_\ell + 4\sigma_0) = 1$ on supp $\phi_{\sigma_0}(||x|^2/t_m^2 - 2(E-\lambda_\ell)| < 2\tau_\ell + 2\sigma_0)$, (7.94) equals

$$\sum_{\ell=1}^{L} \sum_{2\le |d| \le N} \phi_{\sigma_0}^2(\big||x|^2/t_m^2 - 2(E-\lambda_\ell)\big| < 2\tau_\ell + 4\sigma_0)$$
$$\times \phi_{\sigma_0}^2(\big||x|^2/t_m^2 - 2(E-\lambda_\ell)\big| < 2\tau_\ell + 2\sigma_0)\Psi^2(|x|^2/(2t_m^2))\widetilde{P}_d^{M_d^m} e^{-it_m H} f. \quad (7.95)$$

Set

$$B = \langle x\rangle^{-1/2} A \langle x\rangle^{-1/2}, \quad A = \frac{1}{2}(x\cdot p + p\cdot x) = \frac{1}{2}(\langle x, v\rangle + \langle v, x\rangle). \quad (7.96)$$

We note by Theorem 7.3-(7.24) and (7.26) that on the state $\widetilde{P}_d^{M_d^m} e^{-it_m H} f$

$$B \sim \sqrt{2T_d} \sim \frac{|x|}{t_m} \quad (7.97)$$

asymptotically as $t_m \to \infty$. Using this, we replace $\phi_{\sigma_0}(||x|^2/t_m^2 - 2(E-\lambda_\ell)| < 2\tau_\ell + 2\sigma_0)$ by $\phi_{\sigma_0}(|B^2 - 2(E-\lambda_\ell)| < 2\tau_\ell + 2\sigma_0)$ in (7.95). Let $\varphi(\lambda) \in C_0^\infty((\sqrt{2(\lambda_0 - 2\kappa)}, \sqrt{2(\Lambda_0 + 2\kappa)}))$, $0 \le \varphi(\lambda) \le 1$, and $\varphi(\lambda) = 1$ on $[\sqrt{2(\lambda_0-\kappa)}, \sqrt{2(\Lambda_0+\kappa)}] (\supset \text{supp}\, \Psi(\lambda^2/2) \cap (0,\infty))$. We insert a factor $\varphi^2(B)$ into (7.95) and then remove the factor $\Psi^2(|x|^2/(2t_m^2))$ using (7.81):

$$\sum_{\ell=1}^{L} \sum_{2\le |d| \le N} \phi_{\sigma_0}^2(\big||x|^2/t_m^2 - 2(E-\lambda_\ell)\big| < 2\tau_\ell + 4\sigma_0)$$
$$\times \phi_{\sigma_0}^2(\big|B^2 - 2(E-\lambda_\ell)\big| < 2\tau_\ell + 2\sigma_0)\varphi^2(B)\widetilde{P}_d^{M_d^m} e^{-it_m H} f. \quad (7.98)$$

On supp $\phi_{\sigma_0}(||x|^2/t_m^2 - 2(E-\lambda_\ell)| < 2\tau_\ell + 4\sigma_0)$ we have

$$0 < 2(E-\lambda_\ell) - 7\tau_0 \le \frac{|x|^2}{t_m^2} \le 2(E-\lambda_\ell) + 7\tau_0. \quad (7.99)$$

Since (7.76), $|\lambda_\ell - \tilde{\lambda}_\ell| < \tau_0$ and (7.85) imply

$$\frac{2(E-\lambda_\ell)+7\tau_0}{2(E-\lambda_\ell)-7\tau_0} - 1 = \frac{14\tau_0}{2(E-\lambda_\ell)-7\tau_0} < \frac{14\lambda_0''/16}{30\lambda_0/16 - \lambda_0''} < \theta_{N-1}, \quad (7.100)$$

we can apply the partition of unity in Proposition 7.9 to the ring defined by (7.99). Then we obtain

$$\begin{aligned} e^{-it_mH}f \ &\sim \ \sum_{\ell=1}^{L}\sum_{2\le|b|\le N}\sum_{2\le|d|\le N} J_b(x_b/(r_\ell t_m))\phi_{\sigma_0}^2(\left||x|^2/t_m^2-2(E-\lambda_\ell)\right|<2\tau_\ell+4\sigma_0) \\ &\times \ \phi_{\sigma_0}^2(\left|B^2-2(E-\lambda_\ell)\right|<2\tau_\ell+2\sigma_0)\varphi^2(B)\widetilde{P}_d^{M_d^m}e^{-it_mH}f, \end{aligned} \tag{7.101}$$

where

$$r_\ell=\sqrt{2(E-\lambda_\ell)-7\tau_0}>0 \quad (\ell=1,\cdots,L). \tag{7.102}$$

By the property (7.67), only the terms with $d\le b$ remain in (7.101):

$$\begin{aligned} e^{-it_mH}f \ &\sim \ \sum_{\ell=1}^{L}\sum_{2\le|b|\le N}\sum_{d\le b} J_b(x_b/(r_\ell t_m))\phi_{\sigma_0}^2(\left||x|^2/t_m^2-2(E-\lambda_\ell)\right|<2\tau_\ell+4\sigma_0) \\ &\times \ \phi_{\sigma_0}^2(\left|B^2-2(E-\lambda_\ell)\right|<2\tau_\ell+2\sigma_0)\varphi^2(B)\widetilde{P}_d^{M_d^m}e^{-it_mH}f. \end{aligned} \tag{7.103}$$

Using Theorem 7.3-(7.26), we replace x_b/t_m by v_b, and at the same time we introduce a pseudodifferential operator into (7.103):

$$P_b(t)=\phi_\sigma(|x_b/t-v_b|^2<u) \tag{7.104}$$

with $u>0$ sufficiently small. Then (7.103) becomes

$$\begin{aligned} e^{-it_mH}f \ &\sim \ \sum_{\ell=1}^{L}\sum_{2\le|b|\le N}\sum_{d\le b} P_b^2(t_m)J_b(v_b/r_\ell)\phi_{\sigma_0}^2(\left||x|^2/t_m^2-2(E-\lambda_\ell)\right|<2\tau_\ell+4\sigma_0) \\ &\times \ \phi_{\sigma_0}^2(\left|B^2-2(E-\lambda_\ell)\right|<2\tau_\ell+2\sigma_0)\varphi^2(B)\widetilde{P}_d^{M_d^m}e^{-it_mH}f. \end{aligned} \tag{7.105}$$

We rearrange the order of the factors on the RHS of (7.105) using that the factors mutually commute asymptotically as $m\to\infty$ by Theorem 7.3. Setting

$$\begin{aligned} G_{b,\lambda_\ell}(t) \ &= \ P_b(t)\phi_{\sigma_0}(\left||x|^2/t^2-2(E-\lambda_\ell)\right|<2\tau_\ell+4\sigma_0) \\ &\times \ \phi_{\sigma_0}(\left|B^2-2(E-\lambda_\ell)\right|<2\tau_\ell+2\sigma_0)\varphi(B), \end{aligned} \tag{7.106}$$

we obtain

$$e^{-it_mH}f\sim\sum_{\ell=1}^{L}\sum_{2\le|b|\le N}\sum_{d\le b}G_{b,\lambda_\ell}(t_m)^*J_b(v_b/r_\ell)G_{b,\lambda_\ell}(t_m)\widetilde{P}_d^{M_d^m}e^{-it_mH}f. \tag{7.107}$$

Now by some calculus of pseudodifferential operators and Theorem 7.3 we note that $P_b(t)J_b(v_b/r_\ell)$ yields a partition of unity $\tilde{J}_b(x_b/(r_\ell t))$ asymptotically as $m\to\infty$ whose support is close to that of $J_b(x_b/(r_\ell t))$. Then we can recover the terms with $d\not\le b$, and using (7.22), we remove the sum of $\widetilde{P}_d^{M_d^m}$ over $2\le|d|\le N$:

$$e^{-it_mH}f\sim\sum_{\ell=1}^{L}\sum_{2\le|b|\le N}G_{b,\lambda_\ell}(t_m)^*J_b(v_b/r_\ell)G_{b,\lambda_\ell}(t_m)e^{-it_mH}f. \tag{7.108}$$

We note that on the RHS, the support with respect to $B^2/2$ of the derivative $\phi'_{\sigma_0}(|B^2 - 2(E-\lambda_\ell)| < 2\tau_\ell + 2\sigma_0)$ is disjoint with $\Sigma(E)$ by (7.57) and (7.88), and the support of $\varphi'(B)$ is similar by (7.76) and the definition of φ above.

We prove the existence of the limit

$$f_{b,\ell} := \lim_{t\to\infty} e^{itH} G_{b,\lambda_\ell}(t)^* J_b(v_b/r_\ell) G_{b,\lambda_\ell}(t) e^{-itH} f \tag{7.109}$$

for $\ell = 1, \cdots, L$ and b with $2 \le |b| \le N$.

For this purpose we differentiate the function

$$(e^{itH} G_{b,\lambda_\ell}(t)^* J_b(v_b/r_\ell) G_{b,\lambda_\ell}(t) e^{-itH} f, g) \tag{7.110}$$

with respect to t, where $f, g \in E_H(\Delta)\mathcal{H}$. Then writing

$$D_t^b g(t) = i[H_b, g(t)] + \frac{dg}{dt}(t) \tag{7.111}$$

for an operator-valued function $g(t)$, we have

$$\begin{aligned}
&\frac{d}{dt}(e^{itH} G_{b,\lambda_\ell}(t)^* J_b(v_b/r_\ell) G_{b,\lambda_\ell}(t) e^{-itH} f, g)\\
&= (e^{itH} D_t^b(\varphi(B)) \phi_{\sigma_0}(\left|B^2 - 2(E-\lambda_\ell)\right| < 2\tau_\ell + 2\sigma_0)\\
&\quad \times \phi_{\sigma_0}(\left||x|^2/t^2 - 2(E-\lambda_\ell)\right| < 2\tau_\ell + 4\sigma_0) P_b(t) J_b(v_b/r_\ell) G_{b,\lambda_\ell}(t) e^{-itH} f, g)\\
&+ (e^{itH} \varphi(B) D_t^b \left(\phi_{\sigma_0}(\left|B^2 - 2(E-\lambda_\ell)\right| < 2\tau_\ell + 2\sigma_0)\right)\\
&\quad \times \phi_{\sigma_0}(\left||x|^2/t^2 - 2(E-\lambda_\ell)\right| < 2\tau_\ell + 4\sigma_0) P_b(t) J_b(v_b/r_\ell) G_{b,\lambda_\ell}(t) e^{-itH} f, g)\\
&+ (e^{itH} \varphi(B) \phi_{\sigma_0}(\left|B^2 - 2(E-\lambda_\ell)\right| < 2\tau_\ell + 2\sigma_0)\\
&\quad \times D_t^b \left(\phi_{\sigma_0}(\left||x|^2/t^2 - 2(E-\lambda_\ell)\right| < 2\tau_\ell + 4\sigma_0)\right) P_b(t) J_b(v_b/r_\ell) G_{b,\lambda_\ell}(t) e^{-itH} f, g)\\
&+ (e^{itH} \varphi(B) \phi_{\sigma_0}(\left|B^2 - 2(E-\lambda_\ell)\right| < 2\tau_\ell + 2\sigma_0)\\
&\quad \times \phi_{\sigma_0}(\left||x|^2/t^2 - 2(E-\lambda_\ell)\right| < 2\tau_\ell + 4\sigma_0) D_t^b\left(P_b(t)\right) J_b(v_b/r_\ell) G_{b,\lambda_\ell}(t) e^{-itH} f, g)\\
&+ ((h.c.)f, g)\\
&+ (e^{itH} i[I_b, G_{b,\lambda_\ell}(t) J_b(v_b/r_\ell) G_{b,\lambda_\ell}(t)] e^{-itH} f, g),
\end{aligned} \tag{7.112}$$

where $(h.c.)$ denotes the adjoint of the operator in the terms preceding it.

We need the following lemmas (see [24], Lemmas 4.1 and 4.2):

Lemma 7.11 *Let Assumption 7.1 be satisfied. Let $E \in R^1 - \mathcal{T}$. Let $F(s) \in C_0^\infty(R^1)$ satisfy $0 \le F \le 1$ and the condition that the support with respect to $s^2/2$ of $F(s)$ is disjoint with $\Sigma(E)$. Then there is a constant $d(E) > 0$ such that for any interval Δ around E with* diam $\Delta < d(E)$, *one has*

$$\int_{-\infty}^{\infty} \left\| \frac{1}{\sqrt{\langle x \rangle}} F(B) e^{-itH} E_H(\Delta) f \right\|^2 dt \le C\|f\|^2 \tag{7.113}$$

for some constant $C > 0$ independent of $f \in \mathcal{H}$.

Lemma 7.12 *For the pseudodifferential operator $P_b(t)$ defined by (7.104) with $u > 0$, there exist norm continuous bounded operators $S(t)$ and $R(t)$ such that*

$$D_t^b P_b(t) = \frac{1}{t} S(t) + R(t) \tag{7.114}$$

and

$$S(t) \geq 0, \quad \|R(t)\| \leq C\langle t\rangle^{-2} \tag{7.115}$$

for some constant $C > 0$ independent of $t \in R^1$.

We switch to a smaller interval Δ if necessary in the followings when we apply Lemma 7.11.

For the first term on the RHS of (7.112) we have

$$D_t^b(\varphi(B)) = \varphi'(B)i[H_b, B] + R_1 \tag{7.116}$$

with

$$\|(H+i)^{-1}\langle x\rangle^{1/2} i[H_b, B]\langle x\rangle^{1/2}(H+i)^{-1}\| < \infty, \tag{7.117}$$
$$\|(H+i)^{-1}\langle x\rangle R_1 \langle x\rangle (H+i)^{-1}\| < \infty. \tag{7.118}$$

(See section 4 of [24] for a detailed argument yielding the estimates for the remainder terms R_1 here and $S_1(t)$, etc. below.) By the remark after (7.108), the support with respect to $B^2/2$ of $\varphi'(B)$ is disjoint with $\Sigma(E)$. Hence the condition of Lemma 7.11 is satisfied. Thus using (7.117)-(7.118) and rearranging the order of the factors in the first term on the RHS of (7.112) with some integrable errors, we have by Lemma 7.11:

$$\text{the 1st term } = (e^{itH}B_2^{(1)}(t)^* B_1^{(1)}(t)e^{-itH}f, g) + (e^{itH}S_1(t)e^{-itH}f, g), \tag{7.119}$$

where $B_j^{(1)}(t)$ $(j = 1, 2)$ and $S_1(t)$ satisfy

$$\int_{-\infty}^{\infty} \|B_j^{(1)}(t)e^{-itH}f\|^2 dt \leq C\|f\|^2, \tag{7.120}$$
$$\|(H+i)^{-1}S_1(t)(H+i)^{-1}\| \leq Ct^{-2} \tag{7.121}$$

for some constant $C > 0$ independent of $f \in E_H(\Delta)\mathcal{H}$ and $t \in R^1$.

Similarly by another remark after (7.108) and Lemma 7.11, we have a similar bound for the second term on the RHS of (7.112):

$$\text{the 2nd term } = (e^{itH}B_2^{(2)}(t)^* B_1^{(2)}(t)e^{-itH}f, g) + (e^{itH}S_2(t)e^{-itH}f, g), \tag{7.122}$$

where $B_j^{(2)}(t)$ $(j = 1, 2)$ and $S_2(t)$ satisfy

$$\int_{-\infty}^{\infty} \|B_j^{(2)}(t)e^{-itH}f\|^2 dt \leq C\|f\|^2, \tag{7.123}$$
$$\|(H+i)^{-1}S_2(t)(H+i)^{-1}\| \leq Ct^{-2} \tag{7.124}$$

for some constant $C > 0$ independent of $f \in E_H(\Delta)\mathcal{H}$ and $t \in R^1$.

For the third term on the RHS of (7.112), we have

$$\begin{aligned}
&\varphi(B)\phi_{\sigma_0}(\left|B^2 - 2(E-\lambda_\ell)\right| < 2\tau_\ell + 2\sigma_0) \\
&\quad \times D_t^b \left(\phi_{\sigma_0}(\left||x|^2/t^2 - 2(E-\lambda_\ell)\right| < 2\tau_\ell + 4\sigma_0)\right) \\
&= \frac{2}{t}\varphi(B)\phi_{\sigma_0}(\left|B^2 - 2(E-\lambda_\ell)\right| < 2\tau_\ell + 2\sigma_0) \\
&\quad \times \left(\frac{A}{t} - \frac{|x|^2}{t^2}\right)\phi'_{\sigma_0}(\left||x|^2/t^2 - 2(E-\lambda_\ell)\right| < 2\tau_\ell + 4\sigma_0) \\
&\quad + S_3(t),
\end{aligned} \tag{7.125}$$

where $S_3(t)$ satisfies

$$\|(H+i)^{-1}S_3(t)(H+i)^{-1}\| \le Ct^{-2}, \quad t > 1. \tag{7.126}$$

On the support of $\phi'_{\sigma_0}(||x|^2/t^2 - 2(E-\lambda_\ell)| < 2\tau_\ell + 4\sigma_0)$, we have

$$|x|/t \ge \sqrt{2(E-\lambda_\ell) - 2\tau_\ell - 5\sigma_0} > 0 \tag{7.127}$$

by (7.85) and (7.87). Thus there is a large $T > 1$ such that for $t \ge T$ we have $|x| > 1$ and $\langle x\rangle = |x|$, and hence

$$\begin{aligned}
2\left(\frac{A}{t} - \frac{|x|^2}{t^2}\right) &= \frac{\langle x\rangle}{t}\left(\frac{x}{\langle x\rangle}\cdot D_x - \frac{|x|}{t}\right) + \left(D_x \cdot \frac{x}{\langle x\rangle} - \frac{|x|}{t}\right)\frac{\langle x\rangle}{t} \\
&= 2\frac{\langle x\rangle}{t}\left(B - \frac{|x|}{t}\right) + tS_4(t)
\end{aligned} \tag{7.128}$$

with $\|S_4(t)\| \le Ct^{-2}$ for $t \ge T$. By (7.55), we have

$$\operatorname{supp}\phi'_{\sigma_0}(|s| < 2\tau_\ell + 4\sigma_0) \subset I_1 \cup I_2 \tag{7.129}$$

with

$$I_1 = [-2\tau_\ell - 5\sigma_0, -2\tau_\ell - 4\sigma_0], \quad I_2 = [2\tau_\ell + 4\sigma_0, 2\tau_\ell + 5\sigma_0], \tag{7.130}$$

and

$$\phi'_{\sigma_0}(|s| < 2\tau_\ell + 4\sigma_0) \ge 0 \quad \text{for } s \in I_1, \tag{7.131}$$

$$\phi'_{\sigma_0}(|s| < 2\tau_\ell + 4\sigma_0) \le 0 \quad \text{for } s \in I_2. \tag{7.132}$$

Consider the case $|x|^2/t^2 - 2(E-\lambda_\ell) \in I_2$. Then

$$\frac{|x|^2}{t^2} \in [2(E-\lambda_\ell) + 2\tau_\ell + 4\sigma_0, 2(E-\lambda_\ell) + 2\tau_\ell + 5\sigma_0]. \tag{7.133}$$

By the factor $\varphi(B)\phi_{\sigma_0}(|B^2 - 2(E-\lambda_\ell)| < 2\tau_\ell + 2\sigma_0)$, we have

$$B^2 \in [2(E-\lambda_\ell) - 2\tau_\ell - 3\sigma_0, 2(E-\lambda_\ell) + 2\tau_\ell + 3\sigma_0] \tag{7.134}$$

and $B \geq \sqrt{2\lambda_0'} > 0$. Thus

$$B - \frac{|x|}{t} \leq 0. \tag{7.135}$$

Therefore by (7.128) and (7.132), (7.125) is positive in this case up to an integrable error. Similarly we see that (7.125) is positive also in the case $|x|^2/t^2 - 2(E - \lambda_\ell) \in I_1$. Rearranging the order of the factors in the third term on the RHS of (7.112) with an integrable error, we see that it has the form

$$\text{the 3rd term} = (e^{itH}A(t)^*A(t)e^{-itH}f, g) + (e^{itH}S_5(t)e^{-itH}f, g) \tag{7.136}$$

with

$$\|(H+i)^{-1}S_5(t)(H+i)^{-1}\| \leq Ct^{-2}. \tag{7.137}$$

The fourth term on the RHS of (7.112) has a similar form by virtue of Lemma 7.12.

The fifth term $((h.c.)f, f)$ is treated similarly to the terms above.

The sixth term on the RHS of (7.112) satisfies

$$|\text{the 6th term}| \leq Ct^{-1-\min\{\epsilon,\epsilon_1\}}\|f\|\|g\|. \tag{7.138}$$

This estimate follows if we note with using (7.67) and some calculus of pseudodifferential operators as stated after (7.107) that the factor $G_{b,\lambda_\ell}(t)^*J_b(v_b/r_\ell)G_{b,\lambda_\ell}(t)$ restricts the coordinates in the region: $|x_\alpha|^2 > \rho_{|b|}|x|^2/2$.

Summarizing we have proved that (7.112) is written as

$$\begin{aligned}
&\frac{d}{dt}(e^{itH}G_{b,\lambda_\ell}(t)^*J_b(v_b/r_\ell)G_{b,\lambda_\ell}(t)e^{-itH}f, g)\\
&= (e^{itH}A(t)^*A(t)e^{-itH}f, g) + \sum_{k=1}^{2}(e^{itH}B_2^{(k)}(t)^*B_1^{(k)}(t)e^{-itH}f, g)\\
&\qquad +(S_6(t)f, g),
\end{aligned} \tag{7.139}$$

where with some constant $C > 0$ independent of $t > T$ and $f \in \mathcal{H}$

$$\int_T^\infty \|B_j^{(k)}(t)e^{-itH}E_H(\Delta)f\|^2 \leq C\|f\|^2, \quad (j, k = 1, 2) \tag{7.140}$$

$$\|(H+i)^{-1}S_6(t)(H+i)^{-1}\| \leq Ct^{-1-\min\{\epsilon,\epsilon_1\}}. \tag{7.141}$$

Integrating (7.139) with respect to t on an interval $[T_1, T_2] \subset [T, \infty)$, we obtain

$$\begin{aligned}
&(e^{itH}G_{b,\lambda_\ell}(t)^*J_b(v_b/r_\ell)G_{b,\lambda_\ell}(t)e^{-itH}f, g)\Big|_{t=T_1}^{T_2}\\
&= \int_{T_1}^{T_2}(A(t)e^{-itH}f, A(t)e^{-itH}g)dt\\
&\quad +\sum_{k=1}^{2}\int_{T_1}^{T_2}(B_1^{(k)}(t)e^{-itH}f, B_2^{(k)}(t)e^{-itH}g)dt + \int_{T_1}^{T_2}(S_6(t)f, g)dt.
\end{aligned} \tag{7.142}$$

Hence using (7.140), (7.141) and the uniform boundedness of $G_{b,\lambda_\ell}(t)$ in $t > 1$, we have

$$\int_{T_1}^{T_2} \|A(t)e^{-itH}g\|^2 dt \le C\|g\|^2 \tag{7.143}$$

for some constant $C > 0$ independent of $T_2 > T_1 \ge T$ and $g \in E_H(\Delta)\mathcal{H}$.

(7.143) and (7.142) with (7.140) and (7.141) then yield that

$$\left| (e^{itH}G_{b,\lambda_\ell}(t)^* J_b(v_b/r_\ell)G_{b,\lambda_\ell}(t)e^{-itH}f, g)\Big|_{t=T_1}^{T_2} \right| \le \delta(T_1)\|f\|\|g\| \tag{7.144}$$

for some $\delta(T_1) > 0$ with $\delta(T_1) \to 0$ as $T_2 > T_1 \to \infty$. This means that the limit

$$\tilde{f}_b^1 = \lim_{t\to\infty} \sum_{\ell=1}^{L} e^{itH}G_{b,\lambda_\ell}(t)^* J_b(v_b/r_\ell)G_{b,\lambda_\ell}(t)e^{-itH}f \tag{7.145}$$

exists for any $f \in E_H(\Delta)\mathcal{H}$ and b with $2 \le |b| \le N$ if Δ is an interval sufficiently small around $E \in R^1 - \mathcal{T}$: $\operatorname{diam}\Delta < d(E)$. Then the asymptotic decomposition (7.108) implies

$$f = \sum_{2\le|b|\le N} \tilde{f}_b^1 \tag{7.146}$$

for $f = \psi(H)f = E_H(\Delta)f$. Further by the existence of the limit (7.145) and $f = E_H(\Delta)f$, we see that $\tilde{f}_b^1$ satisfies

$$E_H(\Delta)\tilde{f}_b^1 = \tilde{f}_b^1 \tag{7.147}$$

in a way similar to the proof of the intertwining property of wave operators.

Now returning to the first $\Phi(H)f$, and noting that $\operatorname{supp}\Phi$ is compact in $R^1 - \mathcal{T}$, we take a finite number of open intervals $\Delta_{j_0} \subset\subset R^1 - \mathcal{T}$ such that $E_{j_0} \in \Delta_{j_0}$, diam $\Delta_{j_0} < d(E_{j_0})$, and $\operatorname{supp}\Phi \subset\subset \bigcup_{j_0}^{\text{finite}} \Delta_{j_0} \subset\subset R^1 - \mathcal{T}$. Then we can take $\psi_{j_0} \in C_0^\infty(\Delta_{j_0})$ such that $\Phi(H)f = \sum_{j_0}^{\text{finite}} \psi_{j_0}(H)f$. Thus from (7.145)-(7.147), we obtain the existence of the limit for $2 \le |b| \le N$:

$$\tilde{f}_b^1 = \lim_{t\to\infty} \sum_{j_0}^{\text{finite}} \sum_{\ell=1}^{L} e^{itH}G_{b,\lambda_\ell}(t)^* J_b(v_b/r_\ell)G_{b,\lambda_\ell}(t)e^{-itH}\psi_{j_0}(H)f, \tag{7.148}$$

and the relations

$$\Phi(H)f = \sum_{2\le|b|\le N} \tilde{f}_b^1, \quad E_H(\Delta)\tilde{f}_b^1 = \tilde{f}_b^1 \tag{7.149}$$

for any set $\Delta \subset\subset R^1 - \mathcal{T}$ with $\operatorname{supp}\Phi \subset \Delta$.

Set

$$\sigma_j = \sqrt{\gamma^{-1}\rho_j\lambda_0'/2}, \quad \delta_j = \sqrt{\gamma\theta_j\Lambda_0'} \quad (j = 2, 3, \cdots, N, \quad \theta_N = 0). \tag{7.150}$$

Then by (7.148), some calculus of pseudodifferential operators, and

$$\mathrm{supp}\,\big(J_b(x_b/r_\ell)\phi_{\sigma_0}(\big||x|^2-2(E-\lambda_\ell)\big|<2\tau_\ell+4\sigma_0)\big)\subset\subset\tilde{T}_b(\gamma^{-1}\rho_{|b|},\gamma\theta_{|b|}), \qquad (7.151)$$

which follows from (7.61)-(7.63), we see that as $t\to\infty$

$$\begin{aligned}
e^{-itH}\tilde{f}_b^1 &\sim \sum_{k=1}^K\sum_{\ell=1}^L G_{b,\lambda_\ell}(t)^*J_b(v_b/r_\ell)G_{b,\lambda_\ell}(t)e^{-itH}E_H(\Delta_k)f\\
&\sim \prod_{\alpha\not\leq b}F(|x_\alpha|\geq\sigma_{|b|}t)F(|x^b|\leq\delta_{|b|}t)\\
&\qquad\qquad\times\sum_{k=1}^K\sum_{\ell=1}^L G_{b,\lambda_\ell}(t)^*J_b(v_b/r_\ell)G_{b,\lambda_\ell}(t)e^{-itH}E_H(\Delta_k)f\\
&\sim \prod_{\alpha\not\leq b}F(|x_\alpha|\geq\sigma_{|b|}t)F(|x^b|\leq\delta_{|b|}t)e^{-itH}\tilde{f}_b^1. \qquad (7.152)
\end{aligned}$$

(7.149) and (7.152) imply

$$\tilde{f}_b^1\in S_b^{1\sigma_{|b|}\delta_{|b|}}(\Delta)\quad(2\leq|b|\leq N). \qquad (7.153)$$

II) A refinement:

As in (7.86)-(7.87), we take a finite subset $\{\tilde{\lambda}_\ell^b\}_{\ell=1}^{L_b}$ of $\mathcal{T}_b$ for a constant $\tau_0^b>0$ with $\tau_0^b<\tau_0$ such that

$$\mathcal{T}_b\subset\bigcup_{\ell=1}^{L_b}(\tilde{\lambda}_\ell^b-\tau_0^b,\tilde{\lambda}_\ell^b+\tau_0^b), \qquad (7.154)$$

and choose real numbers $\lambda_\ell^b\in R^1$, $\tau_\ell^b>0$ $(\ell=1,\cdots,L_b)$ and $\sigma_0^b>0$ such that

$$\begin{aligned}
&\tau_\ell^b<\tau_0^b,\quad\sigma_0^b<\tau_0^b,\quad|\lambda_\ell^b-\tilde{\lambda}_\ell^b|<\tau_0^b,\\
&\mathcal{T}_b\subset\bigcup_{\ell=1}^{L_b}(\lambda_\ell^b-\tau_\ell^b,\lambda_\ell^b+\tau_\ell^b),\quad(\lambda_\ell^b-\tau_\ell^b,\lambda_\ell^b+\tau_\ell^b)\subset(\tilde{\lambda}_\ell^b-\tau_0^b,\tilde{\lambda}_\ell^b+\tau_0^b),\\
&\mathrm{dist}\{(\lambda_\ell^b-\tau_\ell^b,\lambda_\ell^b+\tau_\ell^b),(\lambda_k^b-\tau_k^b,\lambda_k^b+\tau_k^b)\}>4\sigma_0^b(>0)\quad\text{for any }\ell\neq k. \qquad (7.155)
\end{aligned}$$

We set $\mathcal{T}_b^F=\{\lambda_\ell^b\}_{\ell=1}^{L_b}$ and

$$\tilde{\tau}_0^b=\min_{1\leq\ell\leq L_b}\{\sigma_0^b,\tau_\ell^b\}. \qquad (7.156)$$

Then, we take $\psi_1(\lambda)\in C_0^\infty(R^1)$ such that

$$0\leq\psi_1\leq1, \qquad (7.157)$$

$$\psi_1(\lambda)=\begin{cases}1 & \text{for any }\lambda\text{ with }|\lambda-\lambda_\ell^b|\leq\tilde{\tau}_0^b/2\text{ for some }\lambda_\ell^b\in\mathcal{T}_b^F\\ 0 & \text{for any }\lambda\text{ with }|\lambda-\lambda_\ell^b|\geq\tilde{\tau}_0^b\text{ for all }\lambda_\ell^b\in\mathcal{T}_b^F\end{cases} \qquad (7.158)$$

and we divide (7.148) as follows:

$$\tilde{f}_b^1 = h_b + g_b, \tag{7.159}$$

where

$$h_b = \lim_{t\to\infty} \sum_{j_0}^{\text{finite}} \sum_{\ell=1}^{L} e^{itH} G_{b,\lambda_\ell}(t)^* \psi_1(H^b) J_b(v_b/r_\ell) G_{b,\lambda_\ell}(t) e^{-itH} \psi_{j_0}(H) f, \tag{7.160}$$

$$g_b = \lim_{t\to\infty} \sum_{j_0}^{\text{finite}} \sum_{\ell=1}^{L} e^{itH} G_{b,\lambda_\ell}(t)^* (I-\psi_1)(H^b) J_b(v_b/r_\ell) G_{b,\lambda_\ell}(t) e^{-itH} \psi_{j_0}(H) f. \tag{7.161}$$

The proof of the existence of these limits is similar to that of $\tilde{f}_b^1$ in (7.148), since the change in the present case is the appearance of the commutator $[H, \psi_1(H^b)] = [I_b, \psi(H^b)]$ whose treatment is quite the same as that of the commutators including I_b in (7.138). We introduce the decomposition (7.22) into h_b and g_b on the left of $e^{-itH}\psi_{j_0}(H)f$ as in (7.71). Then by the factor $J_b(v_b/r_\ell)G_{b,\lambda_\ell}(t)$, we see that only the terms with $d \le b$ in the sum in (7.71) remain asymptotically as $t = t_m \to \infty$ by the arguments similar to step I). On each summand $P_{d,E_j}\widehat{P}_{|d|-1}^{\widehat{M}_d^m}$ in these terms (see (7.73)), H^b asymptotically equals $H_d^b = T_d^b + H^d = T_d^b + E_j \sim |x_d^b|^2/(2t_m^2) + E_j \sim |x^b|^2/(2t_m^2) + E_j$, where for $d \le b$, $H_d^b = T_d^b + H^d = H_d - T_b$, $T_d^b = T_d - T_b$ and $x^b = (x_d^b, x^d)$ is a clustered Jacobi coordinate inside the coordinate x^b. Thus we have

$$\begin{aligned} &\psi_1(H^b)J_b(v_b/r_\ell)G_{b,\lambda_\ell}(t)P_{d,E_j}\widehat{P}_{|d|-1}^{\widehat{M}_d^m} e^{-it_mH}\psi_{j_0}(H)f \\ &\sim \psi_1(|x^b|^2/(2t_m^2) + E_j)J_b(v_b/r_\ell)G_{b,\lambda_\ell}(t)P_{d,E_j}\widehat{P}_{|d|-1}^{\widehat{M}_d^m} e^{-it_mH}\psi_{j_0}(H)f \end{aligned} \tag{7.162}$$

as $m \to \infty$. If $\psi_1(|x^b|^2/(2t_m^2) + E_j) \neq 0$, then for some $\ell = 1, \cdots, L_b$

$$\left| \frac{|x^b|^2}{2t_m^2} - (\lambda_\ell^b - E_j) \right| \le \tilde{\tau}_0^b. \tag{7.163}$$

If ℓ is a (unique) $\ell(j)$ such that $E_j \in (\lambda_{\ell(j)}^b - \tau_{\ell(j)}^b, \lambda_{\ell(j)}^b + \tau_{\ell(j)}^b)$, we have

$$\frac{|x^b|^2}{2t_m^2} \le \tau_{\ell(j)}^b + \tilde{\tau}_0^b < \tau_0^b + \tilde{\tau}_0^b. \tag{7.164}$$

Thus setting $\delta' = \sqrt{2(\tau_0^b + \tilde{\tau}_0^b)}$, we have

$$|x^b| \le \delta' t_m. \tag{7.165}$$

If $\ell \neq \ell(j)$, we have by (7.163)

$$0 \le \lambda_\ell^b - E_j + \tilde{\tau}_0^b,$$

from which and (7.155)-(7.156) follows

$$\lambda_\ell^b - E_j \ge 4\sigma_0^b.$$

Thus from (7.163)

$$\frac{|x^b|^2}{2t_m^2} \geq 4\sigma_0^b - \tilde{\tau}_0^b \geq 3\sigma_0^b \geq 3\tilde{\tau}_0^b. \tag{7.166}$$

Setting $\sigma' = \sqrt{6\tilde{\tau}_0^b}$ we then have for $\ell \neq \ell(j)$

$$|x^b| \geq \sigma' t_m. \tag{7.167}$$

Therefore h_b can be decomposed as

$$h_b = f_b^{\delta'} + g_{b1}^{\sigma'}, \tag{7.168}$$

where

$$f_b^{\delta'} = \lim_{t\to\infty} \sum_{j_0}^{\text{finite}} \sum_{\ell=1}^{L} e^{itH} G_{b,\lambda_\ell}(t)^* F(|x^b| \leq \delta' t)\psi_1(H^b) J_b(v_b/r_\ell) G_{b,\lambda_\ell}(t) e^{-itH} \psi_{j_0}(H) f, \tag{7.169}$$

$$g_{b1}^{\sigma'} = \lim_{t\to\infty} \sum_{j_0}^{\text{finite}} \sum_{\ell=1}^{L} e^{itH} G_{b,\lambda_\ell}(t)^* F(|x^b| \geq \sigma' t)\psi_1(H^b) J_b(v_b/r_\ell) G_{b,\lambda_\ell}(t) e^{-itH} \psi_{j_0}(H) f, \tag{7.170}$$

The existence of the limit (7.169) is proved similarly to that of (7.160) by rewriting the factor $F(|x^b| \leq \delta' t)$ as a smooth one and absorbing it into $J_b(v_b/r_\ell)G_{b,\lambda_\ell}(t)$ with changing the constants in it suitably. The existence of (7.170) then follows from this, (7.148) and (7.168).

For g_b, similarly to $g_{b1}^{\sigma'}$ we obtain

$$\begin{aligned} g_b = \lim_{t\to\infty} \sum_{j_0}^{\text{finite}} \sum_{\ell=1}^{L} & e^{itH} G_{b,\lambda_\ell}(t)^* F(|x^b| \geq \sigma' t)(I - \psi_1)(H^b) \\ & \times J_b(v_b/r_\ell) G_{b,\lambda_\ell}(t) e^{-itH} \psi_{j_0}(H) f. \end{aligned} \tag{7.171}$$

Setting

$$\begin{aligned} g_b^{\sigma'} &= g_{b1}^{\sigma'} + g_b \\ &= \lim_{t\to\infty} \sum_{j_0}^{\text{finite}} \sum_{\ell=1}^{L} e^{itH} G_{b,\lambda_\ell}(t)^* F(|x^b| \geq \sigma' t) J_b(v_b/r_\ell) G_{b,\lambda_\ell}(t) e^{-itH} \psi_{j_0}(H) f, \end{aligned} \tag{7.172}$$

we obtain a decomposition of $\tilde{f}_b^1$:

$$\tilde{f}_b^1 = f_b^{\delta'} + g_b^{\sigma'}, \tag{7.173}$$

where $f_b^{\delta'}$ and $g_b^{\sigma'}$ satisfy

$$e^{-itH} f_b^{\delta'} \sim \prod_{\alpha \not\leq b} F(|x_\alpha| \geq \sigma_{|b|} t) F(|x^b| \leq \delta' t) e^{-itH} f_b^{\delta'}, \tag{7.174}$$

$$e^{-itH} g_b^{\sigma'} \sim \prod_{\alpha \not\leq b} F(|x_\alpha| \geq \sigma_{|b|} t) F(|x^b| \leq \delta_{|b|} t) F(|x^b| \geq \sigma' t) e^{-itH} g_b^{\sigma'}. \tag{7.175}$$

We can prove the existence of the limits

$$f_b^1 = \lim_{\delta' \downarrow 0} f_b^{\delta'}, \quad g_b^1 = \lim_{\sigma' \downarrow 0} g_b^{\sigma'} \tag{7.176}$$

in the same way as in Enss [11], Lemma 4.8, because we can take ψ_1 in (7.157)-(7.158) monotonically decreasing when $\tilde{\tau}_0^b \downarrow 0$ and the factors $F(|x^b| \le \delta' t)$ and $F(|x^b| \ge \sigma' t)$ can be treated similarly to ψ_1 by regarding x^b/t as a single variable. Further we have as in (7.147)

$$E_H(\Delta) f_b^1 = f_b^1, \quad E_H(\Delta) g_b^1 = g_b^1, \tag{7.177}$$

which, (7.174) and (7.176) imply

$$f_b^1 \in S_b^1. \tag{7.178}$$

Thus we have a decomposition:

$$\tilde{f}_b^1 = f_b^1 + g_b^1, \quad f_b^1 \in S_b^1. \tag{7.179}$$

$g_b^{\sigma'}$ can be decomposed further by using the partition of unity of the ring $\sigma' \le |x^b|/t \le \delta_{|b|}$ with regarding x^b as a total variable x in Proposition 7.9. Arguing similarly to steps I) and II), we can prove that g_b^1 can be decomposed as a sum of the elements f_d^1 of S_d^1 with $d < b$. Combining this with (7.146), (7.178) and (7.179), we obtain (7.69). □

We remark that Theorem 7.10 implies the asymptotic completeness when the long-range part V_α^L vanishes for all pairs α, because in this case we see straightforwardly that $S_b^1 = \mathcal{R}(W_b^\pm)$, where $W_b^\pm$ are the short-range wave operators defined by

$$W_b^\pm = \text{s-}\lim_{t \to \pm\infty} e^{itH} e^{-itH_b} P_b. \tag{7.180}$$

For the case when long-range part does not vanish, we have the following

Theorem 7.13 *Let Assumptions 7.1 and 7.2 be satisfied. Then*

i) For $2(2+\epsilon)^{-1} < r \le 1$

$$S_b^r = S_b^1. \tag{7.181}$$

ii) If $\epsilon > 2(2+\epsilon)^{-1}$, *i.e. when* $\epsilon > \sqrt{3} - 1$, *we have for all* r *with* $0 \le r \le 1$

$$S_b^r = S_b^1. \tag{7.182}$$

iii) If $\epsilon > 1/2$ *and* $V_\alpha^L(x_\alpha) \ge 0$ *for all pairs* α, *then we have for all* r *with* $0 \le r \le 1$

$$S_b^r = S_b^1. \tag{7.183}$$

Proof: i) and ii) follow from Proposition 5.8 of [6] and Proposition 7.6 above. (7.182) for $r = 0$ follows from the proof of Proposition 5.8 of [6]. iii) follows from Theorem 1.1 and Proposition 4.3 of [25] and (7.186) below: Note that $\mathcal{R}(\Omega_b^\psi)$ in Theorem 1.1-(1.31) of [25] constitutes a dense subset of S_b^1, when ψ varies in $C_0^\infty(R^1 - \mathcal{T})$. □

From Theorem 7.13-ii), iii) and Theorem 7.10 follows

Theorem 7.14 *Let Assumptions 7.1 and 7.2 be satisfied with $\epsilon > 2(2+\epsilon)^{-1}$ or with $\epsilon > 1/2$ and $V_\alpha^L(x_\alpha) \geq 0$ for all pairs α. Then we have for all r with $0 \leq r \leq 1$*

$$\bigoplus_{2\leq|b|\leq N} S_b^r = \mathcal{H}_c(H). \tag{7.184}$$

In the next section, we will construct modified wave operators:

$$W_b^\pm = \text{s-}\lim_{t\to\pm\infty} e^{itH} J_b e^{-itH_b} P_b \tag{7.185}$$

with J_b being an extension of J of [19] to the N-body case. We will then prove

$$\mathcal{R}(W_b^\pm) = S_b^0, \tag{7.186}$$

which and Theorems 7.13 and 7.14 imply

Theorem 7.15 *Let Assumptions 7.1 and 7.2 be satisfied with $\epsilon > 2(2+\epsilon)^{-1}$ or with $\epsilon > 1/2$ and $V_\alpha^L(x_\alpha) \geq 0$ for all pairs α. Then we have for all r with $0 \leq r \leq 1$*

$$\mathcal{R}(W_b^\pm) = S_b^r, \tag{7.187}$$

and

$$\bigoplus_{2\leq|b|\leq N} \mathcal{R}(W_b^\pm) = \mathcal{H}_c(H). \tag{7.188}$$

One might expect that (7.184) and (7.188) are always true, but it is denied:

Theorem 7.16 *Let Assumptions 7.1 and 7.2 be satisfied and let $N \geq 3$. Then the followings hold:*

i) Let $2 \leq |b| \leq N$ and let $E_b(r)$ be the orthogonal projection onto S_b^r $(0 \leq r \leq 1)$. Then $E_b(r_1) \leq E_b(r_2)$ for $0 \leq r_1 \leq r_2 \leq 1$, and the discontinuous points of $E_b(r)$ with respect to $r \in [0,1]$ in the strong operator topology are at most countable.

ii) Let $0 < \epsilon < 1/2$ in Assumption 7.1. Then there are long-range pair potentials $V_\alpha(x_\alpha)$ such that for some cluster decomposition b with $2 \leq |b| \leq N$, $E_b(r)$ is discontinuous at $r = r_0$, where $\epsilon < r_0 := (\epsilon+1)/3 < 1/2$. In particular, there are real numbers r_1 and r_2 with $0 \leq r_1 < r_0 < r_2 \leq 1$ such that

$$S_b^{r_1} \text{ is a proper subset of } S_b^{r_2}. \tag{7.189}$$

Proof: i) By Proposition 7.6, S_b^r $(0 \le r \le 1)$ is a family of closed subspaces of a separable Hilbert space $\mathcal{H}$ that increases when $r \in [0,1]$ increases. Thus the corresponding orthogonal projection $E_b(r)$ $(0 \le r \le 1)$ onto S_b^r increases as r increases, and hence has at most a countable number of discontinuous points with respect to $r \in [0,1]$ in the strong operator topology.

ii) holds by Theorem 4.3 of [49], Theorem 7.10 and Proposition 7.6, for b, $|b| = N-1$, with a suitable choice of pair potentials that satisfy Assumption 7.1. In fact, the sum of the ranges $\mathcal{R}(W_n)$ of Yafaev's wave operators W_n $(n = 1, 2, \cdots)$ in Theorem 4.3 of [49] constitutes a subspace of $(E_b(r_0+0) - E_b(r_0-0))\mathcal{H}$ for b with $|b| = N-1$ by his construction of W_n, which means that $E_b(r)$ is discontinuous at $r = r_0$. Here $E_b(r_0 \pm 0) = \text{s-}\lim_{r\to r_0\pm 0} E_b(r)$. $\square$

7.5 A characterization of the ranges of wave operators

The purpose in this section is to prove relation (7.186) for general long-range pair potentials $V_\alpha(x_\alpha)$ under Assumptions 7.1 and 7.2. The inclusion

$$\mathcal{R}(W_b^{\pm}) \subset S_b^0 \tag{7.190}$$

is a trivial relation for any form of definition of the wave operators $W_b^{\pm}$. Thus our main concern is to prove the reverse inclusion

$$S_b^0 \subset \mathcal{R}(W_b^{\pm}). \tag{7.191}$$

The proof of this inclusion is essentially the same for any definition of wave operators and is not difficult in the light of Enss method [8]. As announced, we here consider the wave operators of the form

$$W_b^{\pm} = \text{s-}\lim_{t\to\pm\infty} e^{itH} J_b e^{-itH_b} P_b, \tag{7.192}$$

where J_b is an extension of the identification operator or stationary modifier introduced in [19] for two-body long-range case. The first task in this section is to construct J_b. Our relation (7.191) then follows from the definition of the scattering spaces S_b^0 and properties of J_b by Enss method.

To make the descriptions simple we hereafter consider the case $V_\alpha^S = 0$ for all pairs α. The recovery of the short-range potentials in the following arguments is easy.

Let a C^∞ function $\chi_0(x)$ of $x \in R^\nu$ satisfy

$$\chi_0(x) = \begin{cases} 1 & (|x| \ge 2) \\ 0 & (|x| \le 1). \end{cases} \tag{7.193}$$

To define J_b we introduce time-dependent potentials $I_{b\rho}(x_b, t)$ for $\rho \in (0,1)$:

$$I_{b\rho}(x_b, t) = I_b(x_b, 0) \prod_{k=1}^{k_b} \chi_0(\rho z_{bk}) \chi_0(\langle \log\langle t\rangle\rangle z_{bk}/\langle t\rangle). \tag{7.194}$$

Then $I_{b\rho}(x_b, t)$ satisfies

$$|\partial_{x_b}^\beta I_{b\rho}(x_b, t)| \le C_\beta \rho^{\epsilon_0} \langle t \rangle^{-\ell} \tag{7.195}$$

for any $\ell \ge 0$ and $0 < \epsilon_0 < \epsilon$ with $\epsilon_0 + \ell < |\beta| + \epsilon$, where $C_\beta > 0$ is a constant independent of t, x_b and ρ.

Then we can apply almost the same arguments as in section 2 of [19] to get a solution $\varphi_b(x_b, \xi_b)$ of the eikonal equation:

$$\frac{1}{2}|\nabla_{x_b}\varphi_b(x_b, \xi_b)|^2 + I_b(x_b, 0) = \frac{1}{2}|\xi_b|^2 \tag{7.196}$$

in some conic region in phase space. More exactly we have the following theorems. Let

$$\cos(z_{bk}, \zeta_{bk}) := \frac{z_{bk} \cdot \zeta_{bk}}{|z_{bk}|_e |\zeta_{bk}|_e},$$

where $|z_{bk}|_e = (z_{bk} \cdot z_{bk})^{1/2}$ is the Euclidean norm. We then set for $R_0, d > 0$ and $\theta \in (0, 1)$

$$\Gamma_\pm(R_0, d, \theta) = \{(x_b, \xi_b) \mid |z_{kb}| \ge R_0, |\zeta_{bk}| \ge d, \pm\cos(z_{bk}, \zeta_{bk}) \ge \theta \ (k = 1, \cdots, k_b)\},$$

where ζ_{bk} is the variable conjugate to z_{bk}.

Theorem 7.17 *Let Assumption 7.1 be satisfied with $V_\alpha^S = 0$ for all pairs α. Then there exists a C^∞ function $\phi_b^\pm(x_b, \xi_b)$ that satisfies the following properties: For any $0 < \theta, d < 1$, there exists a constant $R_0 > 1$ such that for any $(x_b, \xi_b) \in \Gamma_\pm(R_0, d, \theta)$*

$$\frac{1}{2}|\nabla_{x_b}\phi_b^\pm(x_b, \xi_b)|^2 + I_b(x_b, 0) = \frac{1}{2}|\xi_b|^2 \tag{7.197}$$

and

$$|\partial_{x_b}^\alpha \partial_{\xi_b}^\beta (\phi_b^\pm(x_b, \xi_b) - x_b \cdot \xi_b)| \le \begin{cases} C_{\alpha\beta} \left(\max_{1 \le k \le k_b} \langle z_{bk} \rangle\right)^{1-\epsilon}, & \alpha = 0 \\ C_{\alpha\beta} \left(\min_{1 \le k \le k_b} \langle z_{bk} \rangle\right)^{1-\epsilon-|\alpha|}, & \alpha \ne 0, \end{cases} \tag{7.198}$$

where $C_{\alpha\beta} > 0$ is a constant independent of $(x_b, \xi_b) \in \Gamma_\pm(R_0, d, \theta)$.

From this we can derive the following theorem in quite the same way as that for Theorem 2.5 of [19]. Let $0 < \theta < 1$ and let $\psi_\pm(\tau) \in C^\infty([-1, 1])$ satisfy

$$\begin{aligned} 0 &\le \psi_\pm(\tau) \le 1, \\ \psi_+(\tau) &= \begin{cases} 1 & \text{for } \theta \le \tau \le 1, \\ 0 & \text{for } -1 \le \tau \le \theta/2, \end{cases} \\ \psi_-(\tau) &= \begin{cases} 0 & \text{for } -\theta/2 \le \tau \le 1, \\ 1 & \text{for } -1 \le \tau \le -\theta. \end{cases} \end{aligned}$$

We set

$$\chi_\pm(x_b, \xi_b) = \prod_{k=1}^{k_b} \psi_\pm(\cos(z_{bk}, \zeta_{bk}))$$

and define $\varphi_b(x_b,\xi_b)=\varphi_{b,\theta,d,R_0}(x_b,\xi_b)$ by

$$\begin{aligned}\varphi_b(x_b,\xi_b) &= \{(\phi_b^+(x_b,\xi_b)-x_b\cdot\xi_b)\chi_+(x_b,\xi_b)+(\phi_b^-(x_b,\xi_b)-x_b\cdot\xi_b)\chi_-(x_b,\xi_b)\}\\ &\times \prod_{k=1}^{k_b}\chi_0(2\zeta_{bk}/d)\chi_0(2z_{bk}/R_0)+x_b\cdot\xi_b \end{aligned} \tag{7.199}$$

for $d,R_0>0$. Note that $\varphi_{b,\theta,d,R_0}(x_b,\xi_b)=\varphi_{b,\theta,d',R_0'}(x_b,\xi_b)$ when $|z_{bk}|\ge\max(R_0,R_0')$, $|\zeta_{bk}|\ge\max(d,d')$ for all k. We then have

Theorem 7.18 *Let Assumption 7.1 be satisfied with $V_\alpha^S=0$ for all pairs α. Let $0<\theta<1$ and $d>0$. Then there exists a constant $R_0>1$ such that the C^∞ function $\varphi_b(x_b,\xi_b)$ defined above satisfies the following properties.*
i) For $(x_b,\xi_b)\in\Gamma_+(R_0,d,\theta)\cup\Gamma_-(R_0,d,\theta)$, φ_b is a solution of

$$\frac{1}{2}|\nabla_{x_b}\varphi_b(x_b,\xi_b)|^2+I_b(x_b,0)=\frac{1}{2}|\xi_b|^2. \tag{7.200}$$

ii) For any $(x_b,\xi_b)\in R^{2\nu(|b|-1)}$ and multi-indices α,β, φ_b satisfies

$$|\partial_{x_b}^\alpha\partial_{\xi_b}^\beta(\varphi_b(x_b,\xi_b)-x_b\cdot\xi_b)|\le\begin{cases}C_{\alpha\beta}\left(\max\langle z_{bk}\rangle\right)^{1-\epsilon}, & \alpha=0\\ C_{\alpha\beta}\left(\min\langle z_{bk}\rangle\right)^{1-\epsilon-|\alpha|}, & \alpha\ne0.\end{cases} \tag{7.201}$$

In particular, if $\alpha\ne0$,

$$|\partial_{x_b}^\alpha\partial_{\xi_b}^\beta(\varphi_b(x_b,\xi_b)-x_b\cdot\xi_b)|\le C_{\alpha\beta}R_0^{-\epsilon_0}\left(\min\langle z_{bk}\rangle\right)^{1-\epsilon_1-|\alpha|} \tag{7.202}$$

for any $\epsilon_0,\epsilon_1\ge0$ with $\epsilon_0+\epsilon_1=\epsilon$. Further

$$\varphi_b(x_b,\xi_b)=x_b\cdot\xi_b\quad \text{when } |z_{bk}|\le R_0/2 \text{ or } |\zeta_{bk}|\le d/2 \text{ for some } k. \tag{7.203}$$

iii) Let

$$a_b(x_b,\xi_b)=e^{-i\varphi_b(x_b,\xi_b)}\left(T_b+I_b(x_b,0)-\frac{1}{2}|\xi_b|^2\right)e^{i\varphi_b(x_b,\xi_b)}. \tag{7.204}$$

Then

$$a_b(x_b,\xi_b)=\frac{1}{2}|\nabla_{x_b}\varphi_b(x_b,\xi_b)|^2+I_b(x_b,0)-\frac{1}{2}|\xi_b|^2+i(T_b\varphi_b)(x_b,\xi_b) \tag{7.205}$$

and

$$|\partial_{x_b}^\alpha\partial_{\xi_b}^\beta a_b(x_b,\xi_b)|\le\begin{cases}C_{\alpha\beta}\left(\min\langle z_{bk}\rangle\right)^{-1-\epsilon-|\alpha|}, & (x_b,\xi_b)\in\Gamma_+(R_0,d,\theta)\cup\Gamma_-(R_0,d,\theta)\\ C_{\alpha\beta}\left(\min\langle z_{bk}\rangle\right)^{-\epsilon-|\alpha|}\langle\xi_b\rangle, & \text{otherwise.}\end{cases} \tag{7.206}$$

We now define $J_b=J_{b,\theta,d,R_0}$ by

$$J_bf(x_b)=(2\pi)^{-\nu(|b|-1)}\int_{R^{\nu(|b|-1)}}\int_{R^{\nu(|b|-1)}}e^{i(\varphi_b(x_b,\xi_b)-y_b\cdot\xi_b)}f(y_b)dy_bd\xi_b \tag{7.207}$$

for $f \in \mathcal{H}_b = L^2(R^{\nu(|b|-1)})$ as an oscillatory integral (see e.g. [32]). Wave operators $W_b^\pm$ are now defined by

$$W_b^\pm = \text{s-}\lim_{t\to\pm\infty} e^{itH} J_b e^{-itH_b} P_b. \tag{7.208}$$

We note that this definition depends on θ, d, R_0, but applying stationary phase method to e^{-itT_b} in $e^{-itH_b} = e^{-itT_b} \otimes e^{-itH^b}$ on the RHS we see that the dependence disappears in the limit $t \to \pm\infty$ by the remark made just before Theorem 7.18. Further the asymptotic behavior seen by the stationary phase method tells that the inclusion (7.190) holds:

$$\mathcal{R}(W_b^\pm) \subset S_b^0, \tag{7.209}$$

if the limits (7.208) exist. The existence of (7.208) follows from Theorem 7.18-iii), the asymptotic behavior of e^{-itT_b} and Assumptions 7.1-7.2 by noting the relations

$$\begin{aligned}
&(HJ_b - J_bH_b)e^{-itH_b}P_bf(x) = ((T_b + I_b(x_b, x^b))J_b - J_bT_b)e^{-itH_b}P_bf(x),\\
&((T_b + I_b(x_b, 0))J_b - J_bT_b)g(x_b)\\
&\quad = (2\pi)^{-\nu(|b|-1)} \int_{R^{\nu(|b|-1)}} \int_{R^{\nu(|b|-1)}} e^{i(\varphi_b(x_b,\xi_b) - y_b\cdot\xi_b)} a_b(x_b, \xi_b) g(y_b) dy_b d\xi_b
\end{aligned} \tag{7.210}$$

and the fact that $\text{s-}\lim_{M\to\infty} P_b^M = P_b$. Thus to prove the reverse inclusion (7.191)

$$S_b^0 \subset \mathcal{R}(W_b^\pm), \tag{7.211}$$

it suffices to prove that

$$f \in S_b^0 \ominus \mathcal{R}(W_b^\pm) \tag{7.212}$$

implies

$$f = 0. \tag{7.213}$$

To see this we consider the case $t \to +\infty$ and the quantity

$$(I - e^{isH} J_b e^{-isH_b} J_b^{-1}) e^{-itH} f = (J_b - e^{isH} J_b e^{-isH_b}) J_b^{-1} e^{-itH} f \tag{7.214}$$

for $f \in S_b^0(\Delta)$, $\Delta \subset\subset R^1 - \mathcal{T}$, and $t, s \geq 0$ and use Enss method. Here the existence of J_b^{-1} follows from Theorem 3.3 of [23] by taking $R_0 > 0$ in (7.202) large enough (with a slight adaptation to the present case for phases and symbols satisfying the estimates in Theorem 7.18). (7.214) equals

$$-i \int_0^s e^{iuH} (HJ_b - J_bH_b) e^{-iuH_b} J_b^{-1} du\ e^{-itH} f. \tag{7.215}$$

By Definition 7.4-i)-(7.33) of $S_b^0(\Delta)$, we approximate f by $h \in S_b^{0\sigma}(\Delta)$ for some small $\sigma > 0$ with an arbitrarily small error $\delta > 0$ so that $\|f - h\| < \delta$. Then we have for any sufficiently large $R > 0$

$$\limsup_{t\to\infty} \left\| e^{-itH} h - \prod_{\alpha \not\leq b} F(|x_\alpha| \geq \sigma t) F(|x^b| \leq R) e^{-itH} h \right\| < \delta. \tag{7.216}$$

Proposition 7.5-iii) and $h \in S_b^{0\sigma}(\Delta)$ yield that for some sequence $t_m \to \infty$ (as $m \to \infty$)

$$\|(\varphi(x_b/t_m) - \varphi(v_b))e^{-it_mH}h\| \to 0 \quad \text{as} \quad m \to \infty$$

for any $\varphi \in C_0^\infty(R^{\nu(|b|-1)})$. Replacing t and f in (7.215) by t_m and h, we can therefore insert or remove the factor

$$\Phi = \prod_{k=1}^{k_b} Q_k \tilde{F}(|p_{bk}| \geq \sigma')\tilde{F}(|p_b| \leq S)\tilde{F}(|z_{bk}|/t \geq \sigma')\tilde{F}(|x^b| \leq R) \tag{7.217}$$

to or from the left of $e^{-it_mH}h$ in (7.215) anytime with an error $\delta > 0$. Here $p_{bk} = \frac{1}{i}\frac{\partial}{\partial z_{bk}}$, $\sigma' > 0$ is a small number with $\sigma' < \sigma$, $\tilde{F}(|p_b| \leq S)$ comes from $E_H(\Delta)$ in $h = E_H(\Delta)h$, and $\tilde{F}(\tau \leq S)$ is a smooth characteristic function of the set $\{\tau \in R^1 |\ \tau \leq S\}$ with a slope independent of S, and Q_k is a pseudodifferential operator

$$Q_k g(x_b) = (2\pi)^{-\nu(|b|-1)} \int_{R^{\nu(|b|-1)}} \int_{R^{\nu(|b|-1)}} e^{(x_b\cdot\xi_b - y_b\cdot\xi_b)} q_k(z_{bk}, \zeta_{bk}) g(y_b) dy_b d\xi_b \tag{7.218}$$

with symbol $q_k(z_{bk}, \zeta_{bk})$ satisfying

$$\begin{aligned} &|\partial_{z_{bk}}^\beta \partial_{\zeta_{bk}}^\gamma q_k(z_{bk}, \zeta_{bk})| \leq C_{\beta\gamma}\langle z_{bk}\rangle^{-|\beta|}\langle \zeta_{bk}\rangle^{-|\gamma|}, \\ &q_k(z_{bk}, \zeta_{bk}) = 0 \quad \text{for} \quad \cos(z_{bk}, \zeta_{bk}) \leq \theta \text{ or } |z_{bk}| \leq R_0. \end{aligned} \tag{7.219}$$

The order of products in (7.215) of factors in (7.217) and J_b^{-1} may be arbitrary because these factors are mutually commutative asymptotically as $t \to \infty$ by virtue of (7.216). We note that $d > 0$ in the definition of $J_b = J_{b,\theta,d,R_0}$ can be taken smaller than $\sigma' > 0$ beforehand since W_b^+ is independent of $d > 0$ as mentioned. Thus we can assume the following in addition to (7.219):

$$q(z_{bk}, \zeta_{bk}) = 0 \quad \text{for} \quad |\zeta_{bk}| \leq d. \tag{7.220}$$

We now insert the decomposition (7.22) to the left of $e^{-it_mH}h$ in (7.215) with noting $(I - P^{M_1^m})f = f$ by $f \in \mathcal{H}_c(H)$. Then by $\|(I - P^{M_1^m})h - h\| < 2\delta$ and by inserting the factor (7.217) to the left of $e^{-it_mH}h$ after the insertion of (7.22), we have

$$\limsup_{m\to\infty} \|(I - P_b^{M_{|b|}^m})\Phi e^{-it_mH}h\| < 3\delta. \tag{7.221}$$

By the factor $\tilde{F}(|x^b| \leq R)$ in (7.217) and $E_H(\Delta)$ in $h = E_H(\Delta)h$, $P_b^{M_{|b|}^m}$ in (7.221) converges to P_b as $m \to \infty$ in operator norm in the expression (7.221). It thus suffices to consider the quantity

$$\int_0^s e^{iuH}((T_b + I_b(x_b, x^b))J_b - J_bT_b)e^{-iuH_b}J_b^{-1}du\ P_b^{M_{|b|}^{m_0}}\ \Phi e^{-it_mH}h \tag{7.222}$$

for some large but fixed m_0 with an error $\delta > 0$. Since $P_b^{M_{|b|}^{m_0}} = \sum_{j=1}^{M_{|b|}^{m_0}} P_{b,E_j}$ $(0 \leq M_{|b|}^{m_0} < \infty)$ with P_{b,E_j} being one dimensional eigenprojection of H^b corresponding to eigenvalue E_j, (7.222) is reduced to considering

$$\int_0^s e^{-iuE_j}e^{iuH}((T_b + I_b(x_b, x^b))J_b - J_bT_b)P_{b,E_j}e^{-iuT_b}J_b^{-1}\Phi du\ e^{-it_mH}h. \tag{7.223}$$

By Assumptions 7.1-7.2, the factor P_{b,E_j} bounds the variable x^b and yields a short-range error of order $O((\min\langle z_{bk}\rangle)^{-1-\epsilon})$ on the left of e^{-iuT_b} when we replace $I_b(x_b, x^b)$ by $I_b(x_b, 0)$, and we have that (7.223) equals

$$\begin{aligned}\int_0^s e^{-iuE_j} e^{iuH} P_{b,E_j} O(\langle x^b\rangle)((T_b + I_b(x_b, 0))J_b - J_b T_b \quad &+ \quad O((\min\langle z_{bk}\rangle)^{-1-\epsilon})) \\ &\times \quad e^{-iuT_b} J_b^{-1}\Phi du\; e^{-it_m H} h,\end{aligned} \tag{7.224}$$

where $O(\langle x^b\rangle)$ is an operator such that $\langle x^b\rangle^{-1}O(\langle x^b\rangle)$ is bounded. Using (7.210) and the estimate (7.206) in Theorem 7.18-iii) and applying the propagation estimates in Lemma 3.3-ii) of [19] (again with a slight adaptation to the present case), we now get the estimate:

$$\|((T_b + I_b(x_b, 0))J_b - J_b T_b + O((\min\langle z_{bk}\rangle)^{-1-\epsilon}))e^{-iuT_b} J_b^{-1}\Phi\, (\min\langle z_{bk}\rangle)^{\epsilon/2}\,\| \le C\langle u\rangle^{-1-\epsilon/2} \tag{7.225}$$

for some constant $C > 0$ independent of $u \ge 0$. On the other hand (7.216) yields that

$$\|\,(\min\langle z_{bk}\rangle)^{-\epsilon/2}\, e^{-it_m H} h\|$$

is asymptotically less than 2δ as $m \to \infty$. This and (7.225) prove that the norm of (7.222) is asymptotically less than a constant times δ as $m \to \infty$.

Returning to (7.214) we have proved that

$$\begin{aligned}&\limsup_{m\to\infty} \sup_{s\ge 0} \|(I - e^{isH} J_b e^{-isH_b} J_b^{-1})e^{-it_m H} f\| \\ &\approx_\delta \limsup_{m\to\infty} \sup_{s\ge 0} \|(I - e^{isH} J_b e^{-isH_b} J_b^{-1})P_b^{M_{|b|}^m} e^{-it_m H} f\| \le C\delta,\end{aligned} \tag{7.226}$$

where $a \approx_\delta b$ means that $|a - b| \le C\delta$ for some constant $C > 0$. Since wave operator $W_b^+ = \text{s-}\lim_{s\to\infty} e^{isH} J_b e^{-isH_b} P_b$ exists, (7.226) yields

$$\limsup_{m\to\infty} \|(I - W_b^+ J_b^{-1})P_b^{M_{|b|}^m} e^{-it_m H} f\| \le C\delta. \tag{7.227}$$

By the arguments above deriving (7.221) we can remove $P_b^{M_{|b|}^m}$ and get

$$\limsup_{m\to\infty} \|(I - W_b^+ J_b^{-1})e^{-it_m H} f\| \le C\delta. \tag{7.228}$$

Since we assumed (7.212), f is orthogonal to $\mathcal{R}(W_b^+)$. Thus taking the inner product of the vector inside the norm in (7.228) with $e^{-it_m H} f$, we have

$$\|f\|^2 = \lim_{m\to\infty} |(e^{-it_m H} f, e^{-it_m H} f)| = \lim_{m\to\infty} |(e^{-it_m H} f, (I - W_b^+ J_b^{-1})e^{-it_m H} f)| \le C\delta\|f\|.$$

As $\delta > 0$ is arbitrary, this gives $f = 0$, proving (7.213). The proof of (7.186) is complete.

Exercise

1. With $W_\pm$ being defined by (6.39) we consider

$$W_\pm W_\pm^* f$$

for f belonging to a suitable subspace $\mathcal{D}$ of $\mathcal{H}_c(H)$. Show that the operator

$$JJ^* - I$$

defines a compact operator on $\mathcal{D}$, and prove the asymptotic completeness of $W_\pm$ without utilizing the existence of the inverse J^{-1} of J.

We remark that the existence of the inverse is required in section 7.5 as the inverse J_b^{-1} of J_b.

2. Let $\chi(x)$ ($x \in R^1$) be a real-valued C^∞ function with compact support such that $\chi(0) = 1$, and let $g = g(x)$ be a complex-valued C^∞ function with compact support defined on R^1. Set for $t > 0$

$$f(t,x) = \lim_{\epsilon \downarrow 0} \frac{1}{\sqrt{2\pi i t}} \int_{-\infty}^{\infty} e^{i\frac{(x-y)^2}{2t}} g(y)\chi(\epsilon y) dy,$$

where $\arg i = \dfrac{\pi}{2}$. Show the following.

i) For $t > 0$

$$\lim_{\epsilon \downarrow 0} \frac{1}{\sqrt{2\pi i t}} \int_{-\infty}^{\infty} e^{i\frac{(x-y)^2}{2t}} \chi(\epsilon y) dy = 1.$$

ii) For $t > 0$ the following holds uniformly in $x \in R^1$

$$|f(t,x) - g(x)| \le C\sqrt{t},$$

where $C > 0$ is a constant depending only on g.

3. Let f be a bounded, uniformly continuous function from R^1 to $\mathbf{C}$. Let L^1 be the totality of the Lesbegue integrable functions on R^1. For $\varphi \in L^1$, define

$$(\varphi * f)(t) = \int_{-\infty}^{\infty} \varphi(t-r) f(r) dr.$$

Prove for a given $A \in \mathbf{C}$ that the following two conditions 1) and 2) are mutually equivalent.

1) The limit

$$\lim_{t\to\infty} f(t)$$

exists and equals A.

2) For any $\varphi \in L^1$, the limit

$$\lim_{t\to\infty} (\varphi * f)(t)$$

exists and equals

$$A \int_{-\infty}^{\infty} \varphi(r)dr.$$

4. Let $\mathcal{H}$ be a Hilbert space and let H_1, H_2 be selfadjoint operators defined in $\mathcal{H}$ that satisfy the following relation for some bounded operators A_1, A_2 defined on $\mathcal{H}$

$$H_2 = H_1 + A_2^* A_1.$$

Assume that there exist constants $C_j > 0$ $(j = 1, 2)$ such that

$$\int_0^\infty \|A_j e^{-itH_j} f\|^2 dt \leq C_j \|f\|^2 \quad (\forall f \in \mathcal{H},\ j = 1, 2)$$

hold. Then show that for any $f \in \mathcal{H}$, the limit

$$Wf = \lim_{t\to\infty} e^{itH_2} e^{-itH_1} f$$

exists in $\mathcal{H}$ and defines a unitary operator on $\mathcal{H}$.[5]

[5]This is a simplified version of the result of T. Kato, *Wave operators and similarity for some non-selfadjoint operators*, Math. Annalen 162 (1966), 258-279.

Part III

Observation

Chapter 8

Principle of General Relativity

We now see how we can combine relativity and quantum mechanics in our formulation.

We note that the center of mass of a local system $(H_{n\ell}, \mathcal{H}_{n\ell})$ is always at the origin of the space coordinate system $x_{(H_{n\ell},\mathcal{H}_{n\ell})} \in R^3$ for the local system by the requirement: $\sum_{j\in F_{n+1}^{\ell}} m_j X_j = 0$ in Axiom 2.1, and that the space coordinate system describes just the relative motions inside a local system by our formulation. The center of mass of a local system, therefore, cannot be identified from the local system itself, except the fact that it is at the origin of the coordinates.

Moreover, as any two local systems $(H_{n\ell}, \mathcal{H}_{n\ell})$ and $(H_{mk}, \mathcal{H}_{mk})$ are independent mutually in the sense that QM inside $(H_{n\ell}, \mathcal{H}_{n\ell})$ does not affect the QM of another system $(H_{mk}, \mathcal{H}_{mk})$, we see that the time coordinates $t_{(H_{n\ell},\mathcal{H}_{n\ell})}$ and $t_{(H_{mk},\mathcal{H}_{mk})}$, and the space coordinates $x_{(H_{n\ell},\mathcal{H}_{n\ell})} \in R^3$ and $x_{(H_{mk},\mathcal{H}_{mk})} \in R^3$ of these two local systems are independent mutually. Thus the space-time coordinates $(t_{(H_{n\ell},\mathcal{H}_{n\ell})}, x_{(H_{n\ell},\mathcal{H}_{n\ell})})$ and $(t_{(H_{mk},\mathcal{H}_{mk})}, x_{(H_{mk},\mathcal{H}_{mk})})$ are independent between two different local systems $(H_{n\ell}, \mathcal{H}_{n\ell})$ and $(H_{mk}, \mathcal{H}_{mk})$. In particular, insofar as the systems are considered as quantum-mechanical ones, there is no relation between their centers of mass. In other words, the center of mass of any local system cannot be identified by other local systems quantum-mechanically.

Summing these two considerations, we conclude:

(1) The center of mass of a local system $(H_{n\ell}, \mathcal{H}_{n\ell})$ cannot be identified *quantum-mechanically* by any local system $(H_{mk}, \mathcal{H}_{mk})$ including the case $(H_{mk}, \mathcal{H}_{mk}) = (H_{n\ell}, \mathcal{H}_{n\ell})$.

(2) There is no *quantum-mechanical* relation between any two local coordinates $(t_{(H_{n\ell},\mathcal{H}_{n\ell})}, x_{(H_{n\ell},\mathcal{H}_{n\ell})})$ and $(t_{(H_{mk},\mathcal{H}_{mk})}, x_{(H_{mk},\mathcal{H}_{mk})})$ of two different local systems $(H_{n\ell}, \mathcal{H}_{n\ell})$ and $(H_{mk}, \mathcal{H}_{mk})$.

Utilizing these properties of the centers of mass and the coordinates of local systems, we may make any postulates concerning

(1) the motions of the *centers of mass* of various local systems,

and

(2) the relation between two local coordinates of any two local systems.

In particular, we may impose *classical* postulates on them as far as the postulates are consistent in themselves.

Thus we assume an arbitrary but fixed transformation:

$$y_2 = f_{21}(y_1) \tag{8.1}$$

between the coordinate systems $y_j = (y_j^\mu)_{\mu=0}^3 = (y_j^0, y_j^1, y_j^2, y_j^3) = (ct_j, x_j)$ for $j = 1, 2$, where c is the speed of light in vacuum and (t_j, x_j) is the space-time coordinates of the local system $L_j = (H_{n_j\ell_j}, \mathcal{H}_{n_j\ell_j})$. We regard these coordinates $y_j = (ct_j, x_j)$ as *classical* coordinates, when we consider the motions of centers of mass and the relations of coordinates of various local systems. We can now postulate the general principle of relativity on the physics of the centers of mass:

Axiom 8.1 *The laws of physics which control the relative motions of the centers of mass of local systems are covariant under the change of the reference frame coordinates from* $(ct_{(H_{mk},\mathcal{H}_{mk})}, x_{(H_{mk},\mathcal{H}_{mk})})$ *to* $(ct_{(H_{n\ell},\mathcal{H}_{n\ell})}, x_{(H_{n\ell},\mathcal{H}_{n\ell})})$ *for any pair* $(H_{mk}, \mathcal{H}_{mk})$ *and* $(H_{n\ell}, \mathcal{H}_{n\ell})$ *of local systems.*

We note that this axiom is consistent with the Euclidean metric adopted for the quantum-mechanical coordinates inside a local system, because Axiom 8.1 is concerned with classical motions of the centers of mass *outside* local systems, and we are dealing here with a different aspect of nature from the quantum-mechanical one *inside* a local system.

Axiom 8.1 implies the invariance of the distance under the change of coordinates between two local systems. Thus the metric tensor $g_{\mu\nu}(ct, x)$ which appears here satisfies the transformation rule:

$$g^1_{\mu\nu}(y_1) = g^2_{\alpha\beta}(f_{21}(y_1)) \frac{\partial f^\alpha_{21}}{\partial y^\mu_1}(y_1) \frac{\partial f^\beta_{21}}{\partial y^\nu_1}(y_1), \tag{8.2}$$

where $y_1 = (ct_1, x_1)$; $y_2 = f_{21}(y_1)$ is the transformation (8.1) in the above from $y_1 = (ct_1, x_1)$ to $y_2 = (ct_2, x_2)$; and $g^j_{\mu\nu}(y_j)$ is the metric tensor expressed in the classical coordinates $y_j = (ct_j, x_j)$ for $j = 1, 2$.

The second postulate is the principle of equivalence, which asserts that the classical coordinate system $(ct_{(H_{n\ell},\mathcal{H}_{n\ell})}, x_{(H_{n\ell},\mathcal{H}_{n\ell})})$ is a local Lorentz system of coordinates, insofar as it is concerned with the classical behavior of the center of mass of the local system $(H_{n\ell}, \mathcal{H}_{n\ell})$:

Axiom 8.2 *The metric or the gravitational tensor* $g_{\mu\nu}$ *for the center of mass of a local system* $(H_{n\ell}, \mathcal{H}_{n\ell})$ *in the coordinates* $(ct_{(H_{n\ell},\mathcal{H}_{n\ell})}, x_{(H_{n\ell},\mathcal{H}_{n\ell})})$ *of itself are equal to* $\eta_{\mu\nu}$*, where* $\eta_{\mu\nu} = 0$ *for* $\mu \neq \nu$, $= 1$ *for* $\mu = \nu = 1, 2, 3$, *and* $= -1$ *for* $\mu = \nu = 0$.

Since, at the center of mass, the classical space coordinates $x = 0$, Axiom 8.2 together with the transformation rule (8.2) in the above yields

$$g^1_{\mu\nu}(f^{-1}_{21}(ct_2, 0)) = \eta_{\alpha\beta} \frac{\partial f^\alpha_{21}}{\partial y^\mu_1}(f^{-1}_{21}(ct_2, 0)) \frac{\partial f^\beta_{21}}{\partial y^\nu_1}(f^{-1}_{21}(ct_2, 0)). \tag{8.3}$$

Also by the same reason, the relativistic proper time $d\tau = \sqrt{-g_{\mu\nu}(ct,0)dy^\mu dy^\nu} = \sqrt{-\eta_{\mu\nu}dy^\mu dy^\nu}$ at the origin of a local system is equal to c times the quantum-mechanical proper time dt of the system.

By the fact that the classical Axioms 8.1 and 8.2 of physics are imposed on the centers of mass which are uncontrollable quantum-mechanically, and on the relation between the coordinates of different, therefore quantum-mechanically non-related local systems, the consistency of classical relativistic Axioms 8.1 and 8.2 with quantum-mechanical Axioms 2.1 and 2.2 is clear:

Theorem 8.3 *Axioms 2.1, 2.2, 8.1, and 8.2 are consistent.*

For the sake of completeness we state a formal proof of this theorem.

Proof: The local coordinate system $(t_{(H_{n\ell},\mathcal{H}_{n\ell})}, x_{(H_{n\ell},\mathcal{H}_{n\ell})})$ is determined only within each local system $(H_{n\ell},\mathcal{H}_{n\ell})$ by Definition 3.1, through the quantum-mechanical *internal* motions of the system. This coordinate system is independent of the local coordinate system $(t_{(H_{mk},\mathcal{H}_{mk})}, x_{(H_{mk},\mathcal{H}_{mk})})$ of any other local system $(H_{mk},\mathcal{H}_{mk})$. This is due to the mutual independence of the L^2 representations (given by Axiom 2.1) of the base Hilbert spaces $\mathcal{H}_{n\ell}$ and $\mathcal{H}_{mk}$.

The relativity axioms, Axioms 8.1 and 8.2, are concerned merely with the *centers* of mass of local systems $(H_{mk},\mathcal{H}_{mk})$, *observed* by an observer system $(H_{n\ell},\mathcal{H}_{n\ell})$ with coordinate system $(t_{(H_{n\ell},\mathcal{H}_{n\ell})}, x_{(H_{n\ell},\mathcal{H}_{n\ell})})$. This *observer's* coordinate system $(t_{(H_{n\ell},\mathcal{H}_{n\ell})}, x_{(H_{n\ell},\mathcal{H}_{n\ell})})$ is *independent of* the coordinate system $(t_{(H_{mk},\mathcal{H}_{mk})}, x_{(H_{mk},\mathcal{H}_{mk})})$ of the *observed* system $(H_{mk},\mathcal{H}_{mk})$, as stated in the previous paragraph. Because of this independence, the system $(H_{mk},\mathcal{H}_{mk})$ can follow quantum mechanics (Axioms 2.1 and 2.2) *inside* the system *with respect to* its own coordinate system $(t_{(H_{mk},\mathcal{H}_{mk})}, x_{(H_{mk},\mathcal{H}_{mk})})$, *as well as* its *center* of mass can follow general relativity (Axiom 8.1) or any other given postulates *with respect to* the observer's coordinate system $(t_{(H_{n\ell},\mathcal{H}_{n\ell})}, x_{(H_{n\ell},\mathcal{H}_{n\ell})})$. This is the case, even if the coordinate system $(t_{(H_{n\ell},\mathcal{H}_{n\ell})}, x_{(H_{n\ell},\mathcal{H}_{n\ell})})$ of the observer coincides with the coordinate system $(t_{(H_{mk},\mathcal{H}_{mk})}, x_{(H_{mk},\mathcal{H}_{mk})})$ of the observed system itself, because the motion of the *center* of mass and the internal *relative* motion of a local system are mutually independent. Therefore, the local Lorentz postulate (Axiom 8.2) for the *center* of mass of the system $(H_{n\ell},\mathcal{H}_{n\ell})$ also does not contradict the Euclidean postulates in Axioms 2.1 and 2.2 of the internal space-time of that system.

In this sense, Axioms 8.1 and 8.2 are chosen so that the relativity theory holds between the *observed* motions of *centers* of mass of local systems, and *have nothing to do with* the *internal* motion of each local system, which obeys Axioms 2.1 and 2.2. Thus Axioms 8.1 and 8.2 are consistent with Axioms 2.1 and 2.2. □

Chapter 9

Observation

9.1 Preliminaries

Thus far, we did not mention any thoughts about the physics which is actually observed. We have just given two aspects of nature which are mutually independent. We will introduce a procedure which yields what we observe when we see nature. This procedure will not be contradictory with the two aspects of nature which we have discussed, as the procedure is concerned solely with "*how nature looks, at the observer*," i.e. it is solely concerned with "*at the place of the observer, how nature looks*," with some abuse of the word "place." The validity of the procedure should be judged merely through the comparison between the observation and the prediction given by our procedure.

We remark that our approach to observation differs from the traditional approach like canonical quantization of gravity (see e.g., [18]) or from quantum gravity by Ashtekar et al. [2]. They try to "quantize" gravity with regarding the two aspects: quantum physics and gravity as lying on the same level. We do not regard gravity as an actual force or something similar, but we regard it as something 'fictitious' as assumed in axiom 8.2. Nevertheless as we will see below, we can explain some of the relativistic quantum mechanical phenomena.

We note that, in observation, we can observe only a finite number of disjoint systems, say $L_1, \cdots, L_k$ with $k \geq 1$ a finite integer. We cannot grasp an infinite number of systems at a time. Further each system L_j must have only a finite number of elements by the same reason. Thus these systems $L_1, \cdots, L_k$ may be identified with local systems in the sense of the part I and the later chapter 11.

Local systems are quantum-mechanical systems, and their coordinates are confined to their insides insofar as we appeal to Axioms 2.1–2.2. However we postulated Axioms 8.1 and 8.2 on the classical aspects of those coordinates, which make the local coordinates of a local system a classical reference frame for the centers of mass of other local systems. This leaves us the room to define observation as the *classical* observation of the centers of mass of local systems $L_1, \cdots, L_k$. We call this an observation of $L = (L_1, \cdots, L_k)$ inquiring into sub-systems $L_1, \cdots, L_k$, where L is a local system consisting of the particles which belong to one of the local systems $L_1, \cdots, L_k$.

When we observe the sub-local systems $L_1, \cdots, L_k$ of L, we observe the relations or motions among these sub-systems. Internally the local system L behaves following the

Hamiltonian H_L associated to the local system L. However the actual observation differs from what the pure quantum-mechanical calculation gives for the system L. For example, when an electron is scattered by a nucleus with relative velocity close to that of light, the observation is different from the pure quantum-mechanical prediction.

The quantum-mechanical process inside the local system L is described by the evolution

$$\exp(-it_L H_L)f,$$

where f is the initial state of the system and t_L is the local time of the system L. The Hamiltonian H_L is decomposed as follows in virtue of the local Hamiltonians $H_1, \cdots, H_k$, which correspond to the sub-local systems $L_1, \cdots, L_k$:

$$H_L = H^b + T + I, \quad H^b = H_1 + \cdots + H_k.$$

Here $b = (C_1, \cdots, C_k)$ is the cluster decomposition corresponding to the decomposition $L = (L_1, \cdots, L_k)$ of L; $H^b = H_1 + \cdots + H_k$ is the sum of the internal energies H_j inside L_j, and is an operator defined in the internal state space $\mathcal{H}^b = \mathcal{H}_1^b \otimes \cdots \otimes \mathcal{H}_k^b$; $T = T_b$ denotes the intercluster free energy among the clusters $C_1, \cdots, C_k$ defined in the external state space $\mathcal{H}_b$; and $I = I_b = I_b(x) = I_b(x_b, x^b)$ is the sum of the intercluster interactions between various two different clusters in the cluster decomposition b (cf. section 3.2).

The main concern in this process would be the case that the clusters $C_1, \cdots, C_k$ form asymptotically bound states as $t_L \to \infty$, since other cases are hard to be observed along the process when the observer's concern is, as is usually the case in the observation of scattering process, upon the final state of the *bound* sub-systems $L_1, \cdots, L_k$.

The evolution $\exp(-it_L H_L)f$ then behaves asymptotically as $t_L \to \infty$ as follows for some bound states $g_1, \cdots, g_k$ ($g_j \in \mathcal{H}_j^b$) of local Hamiltonians $H_1, \cdots, H_k$ and for some g_0 belonging to the external state space $\mathcal{H}_b$:

$$\exp(-it_L H_L)f \sim \exp(-it_L h_b)g_0 \otimes \exp(-it_L H_1)g_1 \otimes \cdots \otimes \exp(-it_L H_k)g_k, \quad k \geq 1, \tag{9.1}$$

where $h_b = T_b + I_b(x_b, 0)$. It is easy to see that $g = g_0 \otimes g_1 \otimes \cdots \otimes g_k$ is given by

$$g = g_0 \otimes g_1 \otimes \cdots \otimes g_k = \Omega_b^{+*} f = P_b \Omega_b^{+*} f,$$

provided that the decomposition of the evolution $\exp(-it_L H_L)f$ is of the simple form as in (9.1). Here Ω_b^{+*} is the adjoint operator of a canonical wave operator ([6]) corresponding to the cluster decomposition b:

$$\Omega_b^+ = s\text{-}\lim_{t\to\infty} \exp(itH_L) \cdot \exp(-ith_b) \otimes \exp(-itH_1) \otimes \cdots \otimes \exp(-itH_k) P_b,$$

where P_b is the eigenprojection onto the eigenspace of the Hamiltonian $H^b = H_1 + \cdots + H_k$. The process (9.1) just describes the quantum-mechanical process inside the local system L, and does not specify any meaning related with observation up to the present stage.

To see what we observe in actual observations, let us reflect what we observe in scattering process. We note that the observation of scattering processes is concerned with their initial and final stages. At the final stage of observation of scattering processes, the quantities observed are firstly the points hit by the scattered particles on a screen. If the circumstances are properly set up, one can further indicate the momentum of the scattered particles at the final stage to the extent that the uncertainty principle allows. Consider, for example, a scattering process of an electron by a nucleus. Given the magnitude of initial momentum of an electron relative to the nucleus, one can infer the magnitude of momentum of the electron at the final stage to be equal to the initial one by the law of conservation of energy, since the electron is far away from the nucleus at the initial and final stages, and hence the potential energy between them can be neglected compared to the relative kinetic energy. The direction of momentum at the final stage can also be specified, up to the error due to the uncertainty principle, by setting a sequence of slits toward the desired direction at each point on the screen. Then the observer detects only the electrons scattered to that direction. The magnitude of momentum at initial stage can be selected in advance by applying a uniform magnetic field to the electrons, perpendicularly to their momenta, so that they circulate around circles with the radius proportional to the magnitude of momentum, and by setting a sequence of slits that select the desired stream from those electrons. The selection of magnitude of initial momentum makes the direction of momentum ambiguous due to the uncertainty principle, since the sequence of slits lets the position of electrons accurate to some extent. To sum up, the sequences of slits at the initial and final stages necessarily require to take into account the uncertainty principle so that some ambiguity remains in the observation.

However, in the actual observation of a *single* particle, we *have to decide* at which point on the screen the particle hits and which momentum the particle has, using the prepared apparatus like the sequence of slits located at each point on the screen. Even if we impose an interval for the observed values, we *have to assume* that the edges of the interval are sharply designated. These are the assumption which we always impose on what is called "observation." That is to say, we idealize the situation in any observation or in any measurement of a single particle so that the observed values for each particle are sharp for both of the configuration and momentum. In this sense, the values observed actually for each particle must be classical. We have then necessary and sufficient conditions to make predictions about the differential cross section, as we will see in section 9.2.

Summarizing, we observe just the classical quantities for each particle at the final stage of all observations. In other words, even if we cannot know the values actually, we have to *presuppose* that the values observed for each particle have sharp values. We can apply to this fact the remark stated in the fourth paragraph of this section about the possibility of defining observation as that of the *classical* centers of mass of local systems, and may assume that the actually observed values follow the classical Axioms 8.1 and 8.2. Those sharp values actually observed for each particle will give, when summed over the large number of particles, the probabilistic nature of quantum physical phenomena of scattering processes.

Theoretically, the quantum-mechanical, probabilistic nature of scattering processes is described by differential cross section, defined as the square of the absolute value of the scattering amplitude gotten from scattering operators $S_{bd} = W_b^{+*} W_d^-$, where $W_b^{\pm}$ are usual wave operators. Given the magnitude of the initial momentum of the incoming particle and the scattering angle, the differential cross section gives a prediction about the probability at which point and to which direction on the screen each particle hits on the average. However, as we have remarked, the idealized point on the screen hit by each particle and the scattering angle given as an idealized difference between the directions of the initial and final momenta of each particle have sharp values, and the observation at the final stage is *classical*. We are then required to correct these classical observations by taking into account the classical relativistic effects to those classical quantities.

9.2 The first step

As the first step of the relativistic modification of the scattering process, we consider the scattering amplitude $\mathcal{S}(E, \theta)$, where E denotes the energy level of the scattering process and θ is a parameter describing the direction of the scattered particles. Following our remark made in the previous paragraph, we make the following postulate on the scattering amplitude observed in actual experiment:

Axiom 9.1 *When one observes the final stage of scattering phenomena, the total energy E of the scattering process should be regarded as a classical quantity and is replaced by a relativistic quantity, which obeys the relativistic change of coordinates from the scattering system to the observer's system.*

Since it is not known much about $\mathcal{S}(E, \theta)$ in the many body case, we consider an example of the two body case. Consider a scattering phenomenon of an electron by a Coulomb potential Ze^2/r, where Z is a real number, $r = |x|$, and x is the position vector of the electron relative to the scatterer. We assume that the mass of the scatterer is large enough compared to that of the electron and that $|Z|/137 \ll 1$. Then quantum mechanics gives the differential cross section in a Born approximation:

$$\frac{d\sigma}{d\Omega} = |\mathcal{S}(E, \theta)|^2 \approx \frac{Z^2 e^4}{16E^2 \sin^4(\theta/2)},$$

where θ is the scattering angle and E is the total energy of the system of the electron and the scatterer. We assume that the observer is stationary with respect to the center of mass of this system of an electron and the scatterer. Then, since the electron is far away from the scatterer after the scattering and the mass of the scatterer is much larger than that of the electron, we may suppose that the energy E in the formula in the above can be replaced by the *classical* kinetic energy of the electron by Axiom 9.1. Then, assuming that the speed v of the electron relative to the observer is small compared to the speed c of light in vacuum and denoting the rest mass of the electron by m, we have by Axiom 9.1 that E is observed to have the following relativistic value:

$$E' = c\sqrt{p^2 + m^2c^2} - mc^2 = \frac{mc^2}{\sqrt{1 - (v/c)^2}} - mc^2 \approx \frac{mv^2}{2\sqrt{1 - (v/c)^2}},$$

where $p = mv/\sqrt{1 - (v/c)^2}$ is the relativistic momentum of the electron. Thus the differential cross section should be observed approximately equal to

$$\frac{d\sigma}{d\Omega} \approx \frac{Z^2 e^4}{4m^2 v^4 \sin^4(\theta/2)}(1 - (v/c)^2). \tag{9.2}$$

This coincides with the usual relativistic prediction obtained from the Klein-Gordon equation by a Born approximation. See [26], p.297, for a case which involves the spin of the electron.

Before proceeding to the inclusion of gravity in the general k cluster case, we review this two body case. We note that the two body case corresponds to the case $k = 2$, where L_1 and L_2 consist of single particle, therefore the corresponding Hamiltonians H_1 and H_2 are zero operators on $\mathcal{H}^0 = \mathbf{C} =$ the complex numbers. The scattering amplitude $\mathcal{S}(E, \theta)$ in this case is an integral kernel of the scattering matrix $\widehat{S} = \mathcal{F}S\mathcal{F}^{-1}$, where $S = W^{+*}W^-$ is a scattering operator; $W^\pm = s\text{-}\lim_{t\to\pm\infty} \exp(itH_L)\exp(-itT)$ are wave operators (T is negative Laplacian for short-range potentials under an appropriate unit system, while it has to be modified when long-range potentials are included); and $\mathcal{F}$ is Fourier transformation so that $\mathcal{F}T\mathcal{F}^{-1}$ is a multiplication operator by $|\xi|^2$ in the momentum representation $L^2(R^3_\xi)$. By definition, S commutes with T. This makes $\widehat{S}$ decomposable with respect to $|\xi|^2 = \mathcal{F}T\mathcal{F}^{-1}$. Namely, for *a.e.* $E > 0$, there is a unitary operator $\mathcal{S}(E)$ on $L^2(S^2)$, S^2 being two dimensional sphere with radius one, such that for *a.e.* $E > 0$ and $\omega \in S^2$

$$(\widehat{S}h)(\sqrt{E}\omega) = \left(\mathcal{S}(E)h(\sqrt{E}\cdot)\right)(\omega), \quad h \in L^2(R^3_\xi) = L^2((0,\infty), L^2(S^2_\omega), |\xi|^2 d|\xi|).$$

Thus $\widehat{S}$ can be written as $\widehat{S} = \{\mathcal{S}(E)\}_{E>0}$. It is known [20] that $\mathcal{S}(E)$ can be expressed as

$$(\mathcal{S}(E)\varphi)(\theta) = \varphi(\theta) - 2\pi i\sqrt{E}\int_{S^2} \mathcal{S}(E, \theta, \omega)\varphi(\omega)d\omega$$

for $\varphi \in L^2(S^2)$. The integral kernel $\mathcal{S}(E, \theta, \omega)$ with ω being the direction of initial wave, is the scattering amplitude $\mathcal{S}(E, \theta)$ stated in the above and $|\mathcal{S}(E, \theta, \omega)|^2$ is called differential cross section. These are the most important quantities in physics in the sense that they are the *only* quantities which can be observed in actual physical observation.

The energy level E in the previous example thus corresponds to the energy shell $T = E$, and the replacement of E by E' in the above means that T is replaced by a *classical relativistic* quantity $E' = c\sqrt{p^2 + m^2c^2} - mc^2$. We have then seen that the calculation in the above gives a correct relativistic result, which explains the actual observation.

Axiom 9.1 is concerned with the observation of the final stage of scattering phenomena. To include the gravity into our consideration, we extend Axiom 9.1 to the intermediate process of quantum-mechanical evolution. The intermediate process cannot be an object of any *actual* observation, because the intermediate observation would change the process itself, consequently the result observed at the final stage would be altered. Our next Axiom 9.2 is an extension of Axiom 9.1 from the *actual* observation to the *ideal* observation in the sense that Axiom 9.2 is concerned with such invisible intermediate processes and modifies the *ideal* intermediate classical quantities by relativistic change of coordinates. The spirit of the treatment developed below is to trace the quantum-mechanical paths by ideal observations so that the quantities will be transformed into classical quantities at each step, but the quantum-mechanical paths will *not* be altered owing to the *ideality* of the observations. The classical Hamiltonian obtained at the last step will be "requantized" to recapture the quantum-mechanical nature of the process, therefore the ideality of the intermediate observations will be realized in the final expression of the propagator of the observed system.

9.3 The second step

With these remarks in mind, we return to the general k cluster case, and consider a way to include gravity in our framework.

In the scattering process into $k \geq 1$ clusters, what we observe are the centers of mass of those k clusters $C_1, \cdots, C_k$, and of the combined system $L = (L_1, \cdots, L_k)$. In the example of the two body case of section 9.2, only the combined system $L = (L_1, L_2)$ appears due to $H_1 = H_2 = 0$, therefore the replacement of T by E' is concerned with the free energy between two clusters C_1 and C_2 of the combined system $L = (L_1, L_2)$.

Following this treatment of T in the section 9.2, we replace $T = T_b$ in the exponent of $\exp(-it_L h_b) = \exp(-it_L(T_b + I_b(x_b, 0)))$ on the right hand side of the asymptotic relation (9.1) by the relativistic kinetic energy T'_b among the clusters $C_1, \cdots, C_k$ around the center of mass of $L = (L_1, \cdots, L_k)$, defined by

$$T'_b = \sum_{j=1}^{k} \left(c\sqrt{p_j^2 + m_j^2 c^2} - m_j c^2 \right). \tag{9.3}$$

Here $m_j > 0$ is the rest mass of the cluster C_j, which involves all the internal energies like the kinetic energies inside C_j and the rest masses of the particles inside C_j, and p_j is the relativistic momentum of the center of mass of C_j inside L around the center of mass of L. For simplicity, we assume that the center of mass of L is stationary relative to the observer. Then we can set in the exponent of $\exp(-it_L(T'_b + I_b(x_b, 0)))$

$$t_L = t_O, \tag{9.4}$$

where t_O is the observer's time.

For the factors $\exp(-it_L H_j)$ on the right hand side of (9.1), the object of the *ideal* observation is the centers of mass of the k number of clusters $C_1, \cdots, C_k$. These are the ones which now require the relativistic treatment. Since we identify the clusters $C_1, \cdots, C_k$ as their centers of mass moving in a classical fashion, t_L in the exponent of $\exp(-it_L H_j)$ should be replaced by c^{-1} times the classical relativistic proper time at the origin of the local system L_j, which is equal to the quantum-mechanical local time t_j of the sub-local system L_j. By the same reason and by the fact that H_j is the internal energy of the cluster C_j relative to its center of mass, it would be justified to replace the Hamiltonian H_j in the exponent of $\exp(-it_j H_j)$ by the classical relativistic energy *inside* the cluster C_j around its center of mass

$$H'_j = m_j c^2, \tag{9.5}$$

where $m_j > 0$ is the same as in the above.

Summing up, we arrive at the following postulate, which has the same spirit as in Axiom 9.1 and includes Axiom 9.1 as a special case concerned with actual observation:

Axiom 9.2 *In either actual or ideal observation, the space-time coordinates (ct_L, x_L) and the four momentum $p = (p^\mu) = (E_L/c, p_L)$ of the observed system L should be replaced by classical relativistic quantities, which are transformed into the classical quantities (ct_O, x_O)*

and $p = (E_O/c, p_O)$ in the observer's system L_O according to the relativistic change of coordinates specified in Axioms 8.1 and 8.2. Here t_L is the local time of the system L and x_L is the internal space coordinates inside the system L; and E_L is the internal energy of the system L and p_L is the momentum of the center of mass of the system L.

In the case of the present scattering process into k clusters, the system L in this axiom is each of the local systems L_j $(j = 1, 2, \cdots, k)$ and L.

We continue to consider the k centers of mass of the clusters $C_1, \cdots, C_k$. At the final stage of the scattering process, the velocities of the centers of mass of the clusters $C_1, \cdots, C_k$ would be steady, say $v_1, \cdots, v_k$, relative to the observer's system. Thus, according to Axiom 9.2, the local times t_j $(j = 1, 2, \cdots, k)$ in the exponent of $\exp(-it_jH'_j)$, which are equal to c^{-1} times the relativistic proper times at the origins $x_j = 0$ of the local systems L_j, are expressed in the observer's time coordinate t_O by

$$t_j = t_O\sqrt{1 - (v_j/c)^2} \approx t_O\left(1 - v_j^2/(2c^2)\right), \quad j = 1, 2, \cdots, k, \tag{9.6}$$

where we have assumed $|v_j/c| \ll 1$ and used Axioms 8.1 and 8.2 to deduce the Lorentz transformation:

$$t_j = \frac{t_O - (v_j/c^2)x_O}{\sqrt{1 - (v_j/c)^2}}, \quad x_j = \frac{x_O - v_jt_O}{\sqrt{1 - (v_j/c)^2}}.$$

(For simplicity, we wrote the Lorentz transformation for the case of 2-dimensional space-time.)

Inserting (9.3), (9.4), (9.5) and (9.6) into the right-hand side of (9.1), we obtain a classical approximation of the evolution:

$$\exp\left(-it_O[(T'_b + I_b(x_b, 0) + H'_1 + \cdots + H'_k) - (m_1v_1^2/2 + \cdots + m_kv_k^2/2)]\right) \tag{9.7}$$

under the assumption that $|v_j/c| \ll 1$ for all $j = 1, 2, \cdots, k$.

What we want to clarify is the final stage of the scattering process. Thus as we have mentioned, we may assume that all clusters $C_1, \cdots, C_k$ are far away from any of the other clusters and moving almost in steady velocities $v_1, \cdots, v_k$ relative to the observer. We denote by r_{ij} the distance between two centers of mass of the clusters C_i and C_j for $1 \le i < j \le k$. Then, according to our spirit that we are observing the behavior of the centers of mass of the clusters $C_1, \cdots, C_k$ in *classical* fashion following Axioms 8.1 and 8.2, the clusters $C_1, \cdots, C_k$ can be regarded to have gravitation among them. This gravitation can be calculated if we assume Einstein's field equation, $|v_j/c| \ll 1$, and certain conditions that the gravitation is weak (see [36], section 17.4), in addition to our Axioms 8.1 and 8.2. As an approximation of the first order, we obtain the gravitational potential of Newtonian type for, e.g., the pair of the clusters C_1 and $U_1 = \bigcup_{i=2}^k C_i$:

$$-G\sum_{i=2}^{k} m_1m_i/r_{1i},$$

where G is Newton's gravitational constant.

Considering the k body classical problem for the k clusters $C_1, \cdots, C_k$ moving in the sum of these gravitational fields, we see that the sum of the kinetic energies of $C_1, \cdots, C_k$

and the gravitational potentials among them is constant by the classical law of conservation of energy:

$$m_1 v_1^2/2 + \cdots + m_k v_k^2/2 - G \sum_{1\le i<j\le k} m_i m_j / r_{ij} = \text{constant}.$$

Assuming that $v_j \to v_{j\infty}$ as time tends to infinity, we have constant $= m_1 v_{1\infty}^2/2 + \cdots + m_k v_{k\infty}^2/2$. Inserting this relation into (9.7) in the above, we obtain the following as a classical approximation of the evolution (9.1):

$$\exp\left(-it_O\left[T_b' + I_b(x_b,0) + \sum_{j=1}^{k}(m_j c^2 - m_j v_{j\infty}^2/2) - G\sum_{1\le i<j\le k} m_i m_j/r_{ij}\right]\right). \qquad (9.8)$$

What we do at this stage are *ideal* observations, and these observations should not give any sharp classical values. Thus we have to consider (9.8) as a *quantum-mechanical evolution* and we have to recapture the quantum-mechanical feature of the process. To do so we replace p_j in T_b' in (9.8) by a quantum-mechanical momentum D_j, where D_j is a differential operator $-i\frac{\partial}{\partial x_j} = -i\left(\frac{\partial}{\partial x_{j1}}, \frac{\partial}{\partial x_{j2}}, \frac{\partial}{\partial x_{j3}}\right)$ with respect to the 3-dimensional coordinates x_j of the center of mass of the cluster C_j. Thus the actual process should be described by (9.8) with T_b' replaced by a quantum-mechanical Hamiltonian

$$\widetilde{T}_b = \sum_{j=1}^{k}\left(c\sqrt{D_j^2 + m_j^2 c^2} - m_j c^2\right).$$

This procedure may be called "requantization," and is summarized as the following axiom concerning the ideal observation.

Axiom 9.3 *In the expression describing the classical process at the time of the ideal observation, the intercluster momentum $p_j = (p_{j1}, p_{j2}, p_{j3})$ should be replaced by a quantum-mechanical momentum $D_j = -i\left(\frac{\partial}{\partial x_{j1}}, \frac{\partial}{\partial x_{j2}}, \frac{\partial}{\partial x_{j3}}\right)$. Then this gives the evolution describing the intermediate quantum-mechanical process.*

We thus arrive at an approximation for a quantum-mechanical Hamiltonian including gravitational effect up to a constant term, which depends on the system L and its decomposition into $L_1, \cdots, L_k$, but not affecting the quantum-mechanical evolution, therefore can be eliminated:

$$\widetilde{H}_L = \widetilde{T}_b + I_b(x_b,0) - G\sum_{1\le i<j\le k} m_i m_j/r_{ij}$$

$$= \sum_{j=1}^{k}\left(c\sqrt{D_j^2 + m_j^2 c^2} - m_j c^2\right) + I_b(x_b,0) - G\sum_{1\le i<j\le k} m_i m_j/r_{ij}. \qquad (9.9)$$

We remark that the gravitational terms here come from the substitution of local times t_j to the time t_L in the factors $\exp(-it_L H_j)$ on the right-hand side of (9.1). This form of Hamiltonian in (9.9) is actually used in [35] with $I_b = 0$ to explain the stability and instability of cold stars of large mass, showing the effectiveness of the Hamiltonian.

Summarizing these arguments from (9.1) to (9.9), we have obtained the following *interpretation* of the observation of the quantum-mechanical evolution: To get our prediction for the observation of local systems $L_1, \cdots, L_k$, the quantum-mechanical evolution of the combined local system $L = (L_1, \cdots, L_k)$

$$\exp(-it_L H_L)f$$

should be replaced by the following evolution, in the approximation of the first order under the assumption that $|v_j/c| \ll 1$ $(j = 1, 2, \cdots, k)$ and the gravitation is weak,

$$(\exp(-it_O \widetilde{H}_L) \otimes \underbrace{I \otimes \cdots \otimes I}_{k \text{ factors}}) P_b \Omega_b^{+*} f, \tag{9.10}$$

provided that the original evolution $\exp(-it_L H_L)f$ decomposes into k number of clusters $C_1, \cdots, C_k$ as $t_L \to \infty$ in the sense of (9.1). Here b is the cluster decomposition $b = (C_1, \cdots, C_k)$ that corresponds to the decomposition $L = (L_1, \cdots, L_k)$ of L; t_O is the observer's time; and

$$\widetilde{H}_L = \widetilde{T}_b + I_b(x_b, 0) - G \sum_{1 \le i < j \le k} m_i m_j / r_{ij} \tag{9.11}$$

is the relativistic Hamiltonian inside L given by (9.9), which describes the motion of the centers of mass of the clusters $C_1, \cdots, C_k$.

We remark that (9.10) may produce a bound state combining $C_1, \cdots, C_k$ as $t_O \to \infty$ therefore for all t_O, due to the gravitational potentials in the exponent. Note that this is not prohibited by our assumption that $\exp(-it_L H_L)f$ has to decompose into k clusters $C_1, \cdots, C_k$, because the assumption is concerned with the original Hamiltonian H_L but not with the resultant Hamiltonian $\widetilde{H}_L$.

Extending our primitive assumption Axiom 9.1, which was valid for an example stated in section 9.2, we have arrived at a relativistic Hamiltonian $\widetilde{H}_L$, which would describe approximately the intermediate process, under the assumption that the gravitation is weak and the velocities of the particles are small compared to c, by using the Lorentz transformation. We note that, since we started our argument from the asymptotic relation (9.1), which is concerned with the final stage of scattering processes, we could assume that the velocities of particles are almost steady relative to the observer in the correspondent classical expressions of the processes, therefore we could appeal to the Lorentz transformations when performing the change of coordinates in the relevant arguments.

The final values of scattering amplitude should be calculated by using the Hamiltonian $\widetilde{H}_L$. Then they would explain actual observations. This is our prediction for the observation of relativistic quantum-mechanical phenomena including the effects by gravity and quantum-mechanical forces.

In the example discussed in section 9.2, this approach gives the same result as (9.2) in the approximation of the first order, showing the consistency of our spirit (see [27]).

Exercise

Let us consider the light clock which Einstein used in his definition of an ideal clock in special theory of relativity. A light clock consists of two mirrors stood parallel to each other with light running mirrored to each other continuously. The time is then measured as the number of counts that the light hits the mirrors. This clock is placed stationary to an inertial frame of reference, and the time of the frame is defined by the number of the light-hits of this clock. Insofar as the light is considered as a classical wave and the frame is an inertial one which has no acceleration, this clock can measure the time of the frame. One feature of this clock is that the time is defined by utilizing the distance between the two mirrors and the velocity of light in vacuum which is assumed as an absolute constant in special theory of relativity. Thus time is measured only after the distance between the mirrors and the velocity of light are given, and it is not that time measures the motion or velocity of light wave.

This light clock occupies a certain volume in space, and if an acceleration exists, the mirrors in the clock should be distorted according to general theory of relativity. In this case the number of counts of light between two mirrors cannot be regarded as giving the time of a certain definite frame of reference. To any point in the clock, different metrical tensor may be associated, and one cannot determine the time of which point the clock measures. Here appears a problem of the size of the actual clock which cannot be infinitesimally small. In this sense, the operational definition of the clock in general theory of relativity has a problem. It seems that this problem may be avoided by an interpretation that the general theory of relativity is an approximation of reality, and the theory gives a sufficiently good approximation as experiments and astronomical observations show. This problem, however, will be a cause of difficulty when one tries to quantize the field equation of general theory of relativity, for the expected quantized theory should be covariant under the diffeomorphism which transforms a point of a space-time manifold to a point of the manifold, and no point can accommodate any clocks with actual sizes.

To have sound foundations to resolve these problems, we therefore have to find, firstly, a definition of clock and time which should be given through length (or positions) and velocity (or motion) to accord with the spirit of Einstein's light clock, and secondly, our notion of time should have a certain "good" residence just as the inertial frame of reference in special theory of relativity accommodates the light clock.

As a residence, we prepare a Euclidean quantum space, and within that space we have defined a quantum-mechanical clock which measures the common parameter of quantum-mechanical motions of particles in a (local) system consisting of a finite number of particles. Since clocks thus defined are proper to each local system, and local systems are mutually independent as concerns the relation among the coordinates of these systems, we can impose relativistic change of coordinates among them. And the change of coordinates gives a relation among those local systems which would yield relativistic quantum-mechanical Hamiltonians, explaining the actual observations.

These are an outline of what we did in this book. The problem between quantum mechanics and relativistic theories has been noted [3] already in 1932 by John von Neumann in the footnote on page 6 of the English translation [39] of his book "Die Mathematische Grundlagen der Quantenmechanik," Springer-Verlag, Berlin:

– in all attempts to develop a general and relativistic theory of electromag-

> netism, in spite of noteworthy partial successes, the theory (of Quantum Mechanics) seems to lead to great difficulties, which apparently cannot be overcome without the introduction of wholly new ideas.

We below will consider the problem in the case of quantum mechanics and special theory of relativity, and will see a solution in the case of one particle in 3 dimensional space R^3 first. After then we will see a sketchy explanation of general N-particle case.

I. One particle case with mass m

We consider this one particle in the universe expressed as a vector bundle $X \times R^6$ with base space X being the curved Riemannian space where relativistic CM (classical mechanics) holds and with a fibre R^6 (a phase space associated to each point $x \in X$) with an Euclidean structure on which QM (quantum mechanics) holds. Then inside the QM system of the one particle (which we call a "local system" of the particle), the particle follows the Schrödinger equation when the local time t of the system is given as in definition 3.1, and in the classical space outside the local system the particle is observed as a classical particle moving with velocity v relative to the observer in the observer's time.

The motion inside a local system corresponds to the usually conjectured "invisible" *zitterbewegung* of particles like virtual photons associated around elementary particles. This motion exists even when the particle is at "zero-point" energy, e.g. Not only that but the spin, the vortices observed at very low temperatures in superfluidity etc., would be interpreted [37] as this internal motion inside a local system.

Instead of imposing connections among those fibres as is usually expected for the terminology "vector bundle," we postulate a relation between the internal and external worlds by actualizing the "invisible" internal motion as velocity u inside a local system in the following two axioms, which is assumed to hold between the two worlds on the occasion of observation:

A1. Let u be the QM velocity of the particle inside the local system. Then u and v satisfy

$$|u|^2 + |v|^2 = c^2,$$

where c is the velocity of light in vacuum.

A2. The magnitude of momentum inside the local system is observed constant independent of the velocity v relative to the observer:

$$m^2|u|^2 = (m_0)^2c^2,$$

where m_0 is the rest mass of the particle and m is the observed mass of the particle moving with velocity v relative to the observer.

These axioms are an extension of Einstein's principle [7] of the constancy of the velocity of light in vacuum to a principle of the constancy of the velocity of everything when the velocities in the internal quantum mechanical space and the external classical relativistic space are summed. See Natarajan [38] for the natural motivation for these axioms corresponding to postulates IV and V in [38], whereas Natarajan considers both of internal and external worlds are classical. We consider the internal world quantum mechanical. Thus the above axioms need a justification in order for these to have a consistent meaning, which discussion is given below. This will then yield a unification of quantum mechanics and special theory of relativity.

To see the consequences of those axioms, we first consider the internal motion inside the QM local system of the particle.

Let H be the Hamiltonian of the particle inside the QM space:

$$H = -(\hbar^2)/(2m)(\Delta_x) = \frac{1}{2m}\left(\frac{\hbar}{i}\right)^2\left(\frac{\partial}{\partial x}\right)^2 = \frac{1}{2m}P^2,$$

where

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right), \quad \left(\frac{\partial}{\partial x}\right)^2 = \left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 + \left(\frac{\partial}{\partial x_3}\right)^2, \quad P = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

are partial differentiation with respect to 3 dimensional space coordinate $x = (x_1, x_2, x_3)$ and $\hbar = h/(2\pi)$ with h being Planck constant. P is identified with the QM momentum of the particle.

We define a clock of the local system as in definition 3.1. A remark should be added. There we defined a local clock for the particle number $N \geq 2$. The one particle case we are now considering can be regarded in this context a system of 2 particles with the ocnfiguration x above being the relative 3-dimensional coordinates inside the 2 body system. With this understanding, the local time of the system is defined by the t on the exponent of the local clock $\exp(-itH/\hbar)$.

If t is given as such, we can describe the QM motion of the particle by a solution

$$\phi(t) = \exp(-itH/\hbar)\phi(0).$$

of the Schrödinger equation

$$\frac{\hbar}{i}\frac{d}{dt}\phi(t) + H\phi(t) = 0.$$

If we insert the QM velocity P/m into u in postulates A1 and A2, they become

A1. $|P/m|^2 + |v|^2 = c^2$.

A2. $|P|^2 = (m_0)^2c^2$.

But P is a differential operator and the rest are numerical quantities, so these conditions are meaningless. To make these two postulates meaningful, we move to a momentum space by spectral representation or by Fourier transformation as follows.

In our one particle case, we actually considering a two-body system so that there is an interaction term between the two partilces. But for the sake of simplicity we consider the system as free from interaction. Then the solution $\phi(t) = \exp(-itH/\hbar)\phi(0)$ is given by using Fourier transformation $\mathcal{F}$:

$$\mathcal{F}f(p) = (2\pi\hbar)^{-3/2}\int_{R^3}\exp(-ip\cdot x/\hbar)f(x)dx,$$

where $p \in R^3$, $p \cdot x = p_1x_1 + p_2x_2 + p_3x_3$, as follows:

$$\phi(t) = \phi(t,x) = \mathcal{F}^{-1}\exp(-itp^2/(2m\hbar))\mathcal{F}\phi(0).$$

Also the Hamiltonian H is given by

$$H = \frac{1}{2m}\mathcal{F}^{-1}p^2\mathcal{F}.$$

$\mathcal{F}$ is a unitary operator from $\mathcal{H} = L^2(R^3)$ onto itself. Here $L^2(R^3)$ is the Hilbert space of Lebesgue measurable complex-valued functions $f(x)$ on R^3 that satisfies

$$\int_{R^3} |f(x)|^2 dx < \infty,$$

and has inner product and norm:

$$(f,g) = \int_{R^3} f(x)\overline{g(x)}dx, \quad \|f\| = \sqrt{(f,f)},$$

where $\overline{g(x)}$ is the complex conjugate to a complex number $g(x)$.

From this we construct a spectral representation of H as follows:

Let $\mathcal{F}(\lambda)$ $(\lambda > 0)$ be a map from a subspace (exactly speaking, $L^2_s(R^3)$ with $s > 1/2$, see chapter 5) of $L^2(R^3)$ into $L^2(S^2)$ (S^2 is the unit sphere in R^3) defined by

$$\mathcal{F}(\lambda)f(\omega) = (2\lambda)^{1/4}(\mathcal{F}f)(\sqrt{2\lambda}\omega),$$

where ω is in S^2, i.e. $\omega \in R^3$ and $|\omega| = 1$. Then by the above equation we have

$$\mathcal{F}(\lambda)Hf(\omega) = (\lambda/m)\mathcal{F}(\lambda)f(\omega).$$

Thus H is identified with a multiplication by λ/m when transformed by $\mathcal{F}(\lambda)$ into a spectral representation space

$$\widehat{\mathcal{H}} = L^2((0,\infty), L^2(S^2), d\lambda),$$

where $\{\mathcal{F}(\lambda)\}_{\lambda>0}$ is extended to a unitary operator from $\mathcal{H}$ onto $\widehat{\mathcal{H}}$ in the following sense

$$\int_0^\infty \|\mathcal{F}(\lambda)f\|^2_{L^2(S^2)}d\lambda = \|f\|^2_{L^2(R^3)}.$$

(For more details, see chapter 5, section 5.1.)

Summing up we can regard H as a multiplication operator λ/m by moving to a momentum space representation.

Originally, H is

$$H = \frac{1}{2m}P^2,$$

and P is a 3 dimensional QM momentum. Thus formally we have a correspondence

$$\lambda \leftrightarrow P^2/2.$$

Thus if we denote the QM velocity of the particle inside the local system by u, it is

$$u = P/m$$

and satisfies a relation

$$u^2 = P^2/m^2 \leftrightarrow 2\lambda/m^2.$$

Therefore our postulates A1 and A2 above are restated as follows:

A1. $2\lambda/m^2 + |v|^2 = c^2$.

A2. $2\lambda = (m_0)^2c^2$.

These are now meaningful, as the relevant quantities are all numeric. These axioms correspond to a requirement that we think all things in the local system of the one particle, on an energy shell $H = \lambda/m = (m_0)^2c^2/(2m)$ of the Hamiltonian H, whenever considering the observation of the particle.

Show that the follwoing relation holds.

Proposition 1. $m = m_0/\sqrt{1-(v/c)^2}(\geq m_0)$.

We define:

Definition. The period $p(v)$ of a local system moving with velocity v relative to the observer is defined by the relation:

$$p(v)\lambda/(\hbar m) = 2\pi.$$

Thus

$$p(v) = hm/\lambda = 2hm/[(m_0)^2c^2].$$

This gives a period of the local system with the clock on the energy shell $H = \lambda/m$:

$$exp(-itH/\hbar) = exp(-it\lambda/(\hbar m)).$$

In particular, when $v = 0$, the period $p(0)$ takes the minimum value:

$$p(0) = hm_0/\lambda = 2hm_0/[(m_0)^2c^2] = 2h/(m_0c^2).$$

This we call the least period of time (LPT) of the local system. This gives a minimum cycle or period proper to the local system.

The general $p(v)$ is related to this by virtue of the Proposition 1 as follows:

Proposition 2.

$$p(v) = hm/\lambda = p(0)/\sqrt{1-(v/c)^2}(\geq p(0)).$$

This means that the time $p(v)$ that the clock of a local system, moving with velocity v relative to the observer, rounds 1 cycle when it is seen from the observer, is longer than the time $p(0)$ that the observer's clock rounds 1 cycle, and the ratio is given by $p(v)/p(0) = 1/\sqrt{1-(v/c)^2}(\geq 1)$. Thus time, measured by our QM clock, of a local system moving with velocity v relative to the observer becomes slow with the rate $\sqrt{1-(v/c)^2}$, which is exactly the same as the rate that the special theory of relativity gives. This yields that the QM clock obeys the same transformation rule as that for classical relativistic clocks like light clock discussed at the beginning of this exercise, and shows that quantum mechanical clock is equivalent to the relativistic classical clock.

These mean also that the space-time measured by using QM clock defined as the QM evolution of a local system follows the classical relativistic change of coordinates of space-time. Thus giving a consistent unification of QM and special relativistic CM.

As for the validness of the name LPT, we see how it gives the Planck time:

$$t_P = \sqrt{Gh/c^5} = 1.35125 \times 10^{-43} \text{ s},$$

where G is the gravitational constant. In fact, given Planck mass:

$$m_0 = m_P = \sqrt{hc/G} = 5.45604 \times 10^{-5} \text{ g},$$

our LPT yields

$$p(0) = 2h/(\sqrt{hc/G}c^2) = 2\sqrt{hG/c^5} = 2t_P,$$

which is 2 times Planck time.

II. N-particles case with masses m_j $(j = 1, 2, \cdots, N)$

We consider the case after the local system L of N particles are scattered sufficiently. Then the system's solution asymptotically behaves as follows as the system's time $t = t_L$ goes to ∞ (see (9.1)):

$$\exp(-it_L H_L/\hbar)f \sim \exp(-it_L h_b/\hbar)g_0 \otimes \exp(-it_L H_1/\hbar)g_1 \otimes \cdots \otimes \exp(-it_L H_k/\hbar)g_k,$$

where $h_b = T_b + I_b(x_b, 0)$ and $k \geq 1$.

Now for getting the above result also in this case it suffices to note that the Hamiltonians H_ℓ $(\ell = 1, 2, ..., k)$ of each scattered cluster are treated just as in the case I) above but with using a general theorem on the spectral representation of self-adjoint operators H_ℓ. H_ℓ are not necessarily free Hamiltonians and we cannot use the Fourier transformation, but we can use spectral representation theorem so that each H_ℓ is expressed, unitarily equivalently, as λ_ℓ/M_ℓ in some appropriate Hilbert space, where M_ℓ is the mass of the ℓ-th cluster.

Then it is done quite analogously to I) to derive the propositions 1 and 2 above in the present case. Of course these relations depend on ℓ, and show that the dependence of mass and time of each cluster on the relative velocity to the observer is exactly the same as the special theory of relativity gives as in the case I) above.

One of the important consequences of these arguments is that the quantum clock is equal to the classical relativistic clock, which has remained unexplained as one of the greatest mysteries in modern physics in spite of the observed fact that they coincide with high precision.

Part IV

Conclusions

Chapter 10

Inconsistency of Mathematics?

To begin with stating our conclusive thought of this book, we consider that almost all of what we think would be able to be translated into mathematical words insofar as we consider about physical universe as we see in ordinary physical works after Galileo, Descartes, Newton, till the present age. Thus to consider our description of the universe, it would be inevitable to think about metamathematics and set theory which construct the basis of the modern mathematics. In the next chapter, we will try to describe the universe as a contradictory aspect of our language, or exactly speaking, of our mathematical language, in that it could include all sentences as at least meaningful ones so is regarded contradictory as a whole of those meaningful sentences. To be prepared for that purpose, in this chapter we will try to see if mathematics or set theory which is thought to be a basis of modern mathematics is consistent or not. As is well-known, in 1903, Russell's paradoxical set produced immense discussions about the foundation of mathematics, so follows Hilbert's formalism of mathematics, but this direction was negatively answered by Gödel [15]. We will see in this chapter how this theorem of Gödel seems to give a problem that looks telling that mathematics itself is inconsistent.

We consider a formal set theory S, where we can develop a number theory. As no generality is lost, in the following we consider a number theory that can be regarded as a subsystem of S, and will call it $S^{(0)}$.

Definition 10.1 *1) We assume that a Gödel numbering of the system $S^{(0)}$ is given, and denote a formula with the Gödel number n by A_n.*
2) $\mathbf{A}^{(0)}(a,b)$ is a predicate meaning that "a is the Gödel number of a formula A with just one free variable (which we denote by $A(a)$), and b is the Gödel number of a proof of the formula $A(\mathbf{a})$ in $S^{(0)}$," and $\mathbf{B}^{(0)}(a,c)$ is a predicate meaning that "a is the Gödel number of a formula $A(a)$, and c is the Gödel number of a proof of the formula $\neg A(\mathbf{a})$ in $S^{(0)}$." Here $\mathbf{a}$ denotes the formal natural number corresponding to an intuitive natural number a of the meta level.

Definition 10.2 *Let $\mathbf{P}(x_1, \cdots .x_n)$ be an intuitive-theoretic predicate. We say that $\mathbf{P}(x_1, \cdots, x_n)$ is numeralwise expressible in the formal system $S^{(0)}$, if there is a formula $P(x_1, \cdots, x_n)$ with no free variables other than the distinct variables $x_1, \cdots, x_n$ such that, for each particular n-tuple of natural numbers $x_1, \cdots, x_n$, the following holds:*

i) if $\mathbf{P}(x_1, \cdots, x_n)$ *is true, then* $\vdash P(\mathbf{x}_1, \cdots, \mathbf{x}_n)$.

and

ii) if $\mathbf{P}(x_1, \cdots, x_n)$ *is false, then* $\vdash \neg P(\mathbf{x}_1, \cdots, \mathbf{x}_n)$.

Here "true" means "provable on the meta level," and $\vdash P$ means that a formula P is provable in the formal system, e.g., $S^{(0)}$.

Lemma 10.3 *There is a Gödel numbering of the formal objects of the system* $S^{(0)}$ *such that the predicates* $\mathbf{A}^{(0)}(a, b)$ *and* $\mathbf{B}^{(0)}(a, c)$ *defined above are primitive recursive and hence numeralwise expressible in* $S^{(0)}$ *with the associated formulas* $A^{(0)}(a, b)$ *and* $B^{(0)}(a, c)$. *(See [33].)*

Definition 10.4 *Let* $q^{(0)}$ *be the Gödel number of a formula:*

$$\forall b[\neg A^{(0)}(a, b) \vee \exists c(c \leq b \,\&\, B^{(0)}(a, c))].$$

Namely

$$A_{q^{(0)}}(a) = \forall b[\neg A^{(0)}(a, b) \vee \exists c(c \leq b \,\&\, B^{(0)}(a, c))].$$

In particular

$$A_{q^{(0)}}(\mathbf{q}^{(0)}) = \forall b[\neg A^{(0)}(\mathbf{q}^{(0)}, b) \vee \exists c(c \leq b \,\&\, B^{(0)}(\mathbf{q}^{(0)}, c))].$$

Assume that $S^{(0)}$ is consistent.

Suppose that

$$\vdash A_{q^{(0)}}(\mathbf{q}^{(0)}) \text{ in } S^{(0)},$$

and let $k^{(0)}$ be the Gödel number of the proof of $A_{q^{(0)}}(\mathbf{q}^{(0)})$. Then by the numeralwise expressibility of $\mathbf{A}^{(0)}(a, b)$

$$\vdash A^{(0)}(\mathbf{q}^{(0)}, \mathbf{k}^{(0)}). \tag{10.1}$$

Under our hypothesis of consistency,

$$\vdash A_{q^{(0)}}(\mathbf{q}^{(0)}) \text{ in } S^{(0)}$$

implies

$$\text{not } \vdash \neg A_{q^{(0)}}(\mathbf{q}^{(0)}) \text{ in } S^{(0)}.$$

Hence, for any integer ℓ, $\mathbf{B}^{(0)}(q^{(0)}, \ell)$ is false. In particular, $\mathbf{B}^{(0)}(q^{(0)}, 0), \cdots, \mathbf{B}^{(0)}(q^{(0)}, k^{(0)})$ are false. By virtue of the numeralwise expressibility of $\mathbf{B}^{(0)}(a, c)$, from these follows that

$$\vdash \neg B^{(0)}(\mathbf{q}^{(0)}, 0), \vdash \neg B^{(0)}(\mathbf{q}^{(0)}, 1), \cdots, \vdash \neg B^{(0)}(\mathbf{q}^{(0)}, \mathbf{k}^{(0)}).$$

Hence

$$\vdash \forall c(c \leq \mathbf{k}^{(0)} \supset \neg B^{(0)}(\mathbf{q}^{(0)}, c)).$$

This together with $\vdash A^{(0)}(\mathbf{q}^{(0)}, \mathbf{k}^{(0)})$ in (10.1) gives

$$\vdash \exists b[A^{(0)}(\mathbf{q}^{(0)}, b) \,\&\, \forall c(c \leq b \supset \neg B^{(0)}(\mathbf{q}^{(0)}, c))].$$

This is equivalent to

$$\vdash \neg A_{q^{(0)}}(\mathbf{q}^{(0)}) \text{ in } S^{(0)}.$$

A contradiction with our consistency hypothesis of $S^{(0)}$. Thus

$$\text{not } \vdash A_{q^{(0)}}(\mathbf{q}^{(0)}) \text{ in } S^{(0)}.$$

Reversely, suppose that

$$\vdash \neg A_{q^{(0)}}(\mathbf{q}^{(0)}) \text{ in } S^{(0)}.$$

Then there is a Gödel number $k^{(0)}$ of the proof of $\neg A_{q^{(0)}}(\mathbf{q}^{(0)})$ in $S^{(0)}$, and we have

$$\mathbf{B}^{(0)}(q^{(0)}, k^{(0)}) \text{ is true.}$$

Thus

$$\vdash B^{(0)}(\mathbf{q}^{(0)}, \mathbf{k}^{(0)}),$$

from which follows

$$\vdash \forall b[b \geq \mathbf{k}^{(0)} \supset \exists c(c \leq b \,\&\, B^{(0)}(\mathbf{q}^{(0)}, c))]. \tag{10.2}$$

As $\neg A_{q^{(0)}}(\mathbf{q}^{(0)})$ is provable in $S^{(0)}$, there is no proof of $A_{q^{(0)}}(\mathbf{q}^{(0)})$ in $S^{(0)}$ by our consistency assumption of $S^{(0)}$. Therefore

$$\vdash \neg A^{(0)}(\mathbf{q}^{(0)}, 0), \vdash \neg A^{(1)}(\mathbf{q}^{(1)}, 1), \cdots, \vdash \neg A^{(0)}(\mathbf{q}^{(0)}, \mathbf{k}^{(0)} - 1)$$

hold. Thus

$$\vdash \forall b[b < \mathbf{k}^{(0)} \supset \neg A^{(0)}(\mathbf{q}^{(0)}, b)]. \tag{10.3}$$

Combining (10.2) and (10.3), we obtain

$$\vdash \forall b[\neg A^{(0)}(\mathbf{q}^{(0)}, b) \vee \exists c(c \leq b \,\&\, B^{(0)}(\mathbf{q}^{(0)}, c))],$$

which is

$$\vdash A_{q^{(0)}}(\mathbf{q}^{(0)}).$$

A contradiction with our consistency assumption of $S^{(0)}$. Thus we have

$$\text{not } \vdash \neg A_{q^{(0)}}(\mathbf{q}^{(0)}) \text{ in } S^{(0)}.$$

We have reproduced Rosser's form of Gödel incompleteness theorem.

Lemma 10.5 *Assume $S^{(0)}$ is consistent. Then neither $A_{q^{(0)}}(\mathbf{q}^{(0)})$ nor $\neg A_{q^{(0)}}(\mathbf{q}^{(0)})$ is provable in $S^{(0)}$.*

Thus, we can add either one of $A_{q^{(0)}}(\mathbf{q}^{(0)})$ or $\neg A_{q^{(0)}}(\mathbf{q}^{(0)})$, which we will denote $A_{(0)}$ hereafter, as a new axiom of $S^{(0)}$ without introducing any contradiction. Namely, let $S^{(1)}$ be an extension of the formal system $S^{(0)}$ with an additional axiom $A_{(0)}$. Then by Lemma 10.5

$$S^{(1)} \text{ is consistent.} \tag{10.4}$$

We now extend definitions 10.1 and 10.4 to the extended system $S^{(1)}$ as follows with noting that the numeralwise expressibility of the predicates $\mathbf{A}^{(1)}(a,b)$ and $\mathbf{B}^{(1)}(a,c)$ defined below can be extended to the new system $S^{(1)}$ with *the same* Gödel numbering *as* the one given in Lemma 10.3 for $S^{(0)}$.

1) $\mathbf{A}^{(1)}(a,b)$ is a predicate meaning that "a is the Gödel number of a formula $A(a)$, and b is the Gödel number of a proof of the formula $A(\mathbf{a})$ in $S^{(1)}$," and $\mathbf{B}^{(1)}(a,c)$ is a predicate meaning that "a is the Gödel number of a formula $A(a)$, and c is the Gödel number of a proof of the formula $\neg A(\mathbf{a})$ in $S^{(1)}$."

2) Let $q^{(1)}$ be the Gödel number of a formula:

$$\forall b[\neg A^{(1)}(a,b) \vee \exists c(c \leq b \,\&\, B^{(1)}(a,c))].$$

By the extended numeralwise expressibility, we have in the way similar to Lemma 10.5 by using the consistency (10.4) of $S^{(1)}$

$$\text{not } \vdash A_{q^{(1)}}(\mathbf{q}^{(1)}) \quad \text{and} \quad \text{not } \vdash \neg A_{q^{(1)}}(\mathbf{q}^{(1)}) \quad \text{in } S^{(1)}.$$

Continuing the similar procedure, we get for any natural number $n (\geq 0)$ that

$$S^{(n)} \text{ is consistent,} \tag{10.5}$$

$$\text{not } \vdash A_{q^{(n)}}(\mathbf{q}^{(n)}) \quad \text{and} \quad \text{not } \vdash \neg A_{q^{(n)}}(\mathbf{q}^{(n)}) \quad \text{in } S^{(n)}. \tag{10.6}$$

We now let $S^{(\omega)}$ the extended system of $S^{(0)}$ that includes all of the formulas $A_{(n)} (= A_{q^{(n)}}(\mathbf{q}^{(n)})$ or $\neg A_{q^{(n)}}(\mathbf{q}^{(n)}))$ as its axioms. By (10.5) $S^{(\omega)}$ is consistent. We note that the formula $A_{(n)}$ is recursively defined if we have already constructed the system $S^{(n)}$. Moreover, if we let $\tilde{q}(n)$ be the Gödel number of the formula $A_{(n)}$, we have $\tilde{q}(i) \neq \tilde{q}(j)$ for all natural numbers $i < j$ as $A_{(j)}$ is not provable in $S^{(i+1)}$ for $i < j$. Thus $\sup_{i \leq n} \tilde{q}(i)$ goes to infinity as n tends to infinity. Further we note that $\tilde{q}(n)$ is a recursive function of n. Then given a formula A_r with Gödel number r, restricting our attention to the formulas $A_{(n)}$ with $\tilde{q}(n) \leq r$, we can determine in $S^{(\omega)}$ recursively if that given formula A_r is an axiom of the form $A_{(n)}$ or not. Thus the addition of all $A_{(n)}$ retains the recursive definition of the following predicates $\mathbf{A}^{(\omega)}(a,b)$ and $\mathbf{B}^{(\omega)}(a,c)$ defined in the same way as above.

$\mathbf{A}^{(\omega)}(a,b)$ is a predicate meaning that "a is the Gödel number of a formula $A(a)$, and b is the Gödel number of a proof of the formula $A(\mathbf{a})$ in $S^{(\omega)}$," and $\mathbf{B}^{(\omega)}(a,c)$ is a predicate meaning that "a is the Gödel number of a formula $A(a)$, and c is the Gödel number of a proof of the formula $\neg A(\mathbf{a})$ in $S^{(\omega)}$."

Then we see that the predicates $\mathbf{A}^{(\omega)}(a,b)$ and $\mathbf{B}^{(\omega)}(a,c)$ are numeralwise expressible in $S^{(\omega)}$ and the Gödel number $q^{(\omega)}$ of the formula:

$$\forall b[\neg A^{(\omega)}(a,b) \vee \exists c(c \leq b \,\&\, B^{(\omega)}(a,c))],$$

denoted by $A_{q^{(\omega)}}(a)$, is well-defined.

As before, we continue the similar procedure, transfinite inductively. In this process, from the nature of our extension procedure, at each step α where we construct the α-th consistent system $S^{(\alpha)}$ from the preceding systems $S^{(\gamma)}$ with $\gamma < \alpha$, the predicates $\mathbf{A}^{(\alpha)}(a,b)$ and $\mathbf{B}^{(\alpha)}(a,c)$ must be recursively defined from the preceding predicates $\mathbf{A}^{(\gamma)}(a,b)$ and $\mathbf{B}^{(\gamma)}(a,c)$ $(\gamma < \alpha)$ so as for $S^{(\alpha)}$ to be further extended with retaining consistency. For this to hold, it is necessary and sufficient that the ordinal α is a recursive ordinal.

Is there any ordinal that is not recursive? A function $F(x)$ is called recursive if it has the form:

$$F(x) = G(x, F|x),$$

where $F|x$ is a restriction of F to a domain x, and G is a given function. Consider the formula for an ordinal x:

$$x = \{y | y \in x\}.$$

This meets the above requirement of the recursive definition of the ordinal x, although this is tautological and may not be considered a definition usually. But if we see the structure that it contains the domain x and by using that domain only it defines x itself, it is not so unreasonable to think that there is no nonrecursive ordinal.

There is, however, a possibility ([17]) that the condition whether or not a nonrecursive ordinal exists in ZFC is independent of the axioms of ZFC. In that case we have two alternatives.

Case i) There is no nonrecursive ordinal, and hence all ordinals are recursive.

In this case, the extension of the system $S^{(\alpha)}$ above is always possible. Thus we can extend $S^{(\alpha)}$ indefinitely forever. However, in this process, we cannot reach the step where the number of added axioms is the cardinality $\aleph_1$ of the first uncountable ordinal, as the number of added axioms is at most countable by the nature of formal system. Thus there must be a least countable ordinal β such that the already constructed consistent system $S^{(\beta)}$ is not extendable with retaining consistency. This contradicts the unlimited extendibility stated above, and we have a contradiction. Insofar as we assume that every ordinal is recursive, the only possibility remaining is to conclude that set theory is inconsistent.

Case ii) There is a nonrecursive ordinal, thus there is a least nonrecursive ordinal ω_1 usually called Church-Kleene ordinal ([12], [47]).

In this case the above extension of $S^{(\alpha)}$ is possible if and only if $\alpha < \omega_1$. We note that ω_1 is a limit ordinal. For if it is a successor of an ordinal δ, then $\delta < \omega_1$ is recursive, hence so is $\omega_1 = \delta + 1$, a contradiction with the nonrecursiveness of ω_1. Therefore we can construct, in the same way as that for $S^{(\omega)}$, a consistent system $S^{(\omega_1)}$, which cannot be extended further with retaining consistency by the nonrecursiveness of ω_1.

On the other hand, as we have seen in the discussion of case i), there must be a least countable ordinal β such that the already constructed consistent system $S^{(\beta)}$ is not extendable with retaining consistency. Since β is the least ordinal such that $S^{(\beta)}$ is not extendable, for any $\alpha < \beta$ the system $S^{(\alpha)}$ is consistently extendable. Whence by the reasoning above about the recursiveness of α with which $S^{(\alpha)}$ is consistently extendable, α is recursive and we have $\alpha < \omega_1$ if $\alpha < \beta$. Thus

$$\beta \leq \omega_1. \tag{10.7}$$

Reversely, when $\alpha < \omega_1$, α is a recursive ordinal. Thus by the same reasoning as above about the recursiveness of α, $S^{(\alpha)}$ is consistently extendable. Therefore $\alpha < \beta$ if $\alpha < \omega_1$. This and (10.7) give

$$\beta = \omega_1.$$

Summarizing, we have proved

Theorem 10.6 *Assume that $S^{(0)}$ is consistent. Suppose that the condition whether or not there is a nonrecursive ordinal is independent of the axioms of ZFC. Then there are the following two alternatives:*

i) There is no nonrecursive ordinal, and hence all ordinals are recursive.

In this case set theory is inconsistent.

ii) There is a nonrecursive ordinal, thus there is a least countable nonrecursive ordinal β.

In this case the corresponding system $S^{(\beta)}$ is consistent and cannot be extended further with retaining consistency.

We remark that this is a metamathematical theorem.

Thus the inconsistency in i) of this theorem does not give any proof in ZFC of the existence of nonrecursive ordinal. To know whether a nonrecursive ordinal exists or not, we need a proof in ZFC or if such a statement is independent of the axioms of ZFC, we need to add an axiom that determines which the case is. In the latter case, the above theorem shows a direction in which the extended ZFC can be consistent if the original ZFC is consistent.

Further, as the above theorem is a metamathematical theorem, even if there is no nonrecursive ordinal, the case i) of the theorem does not yield that set theory is inconsistent in the sense that we can find a concrete inconsistent proposition like Russell's paradox inside the set theory. Rather it would be said that we may not find such an inconsistent proposition insofar as we work inside the set theory ZFC. Thus this theorem should not be interpreted as stating any concrete inconsistency of set theory.

Chapter 11

Stationary Universe

By nature what is called the universe must be a closed universe, within which is all. We will characterize it by a certain quantum-mechanical condition.

We consider a metatheory of a formal set theory S. We name this metatheory M_S, indicating that it is a Meta-theory of S as well as a Meta-Scientific theory as Ronald Swan [46] refers to. The following arguments are all made in M_S.

We define in M_S

$$\phi = \text{ the class of all well-formed formulae of } S.$$

This ϕ is a countable set in the context of M_S.

We identify ϕ with the set of truth values (in complex numbers $\mathbf{C}$) of well-formed formulae (wff's) in ϕ. In this identification, we define a map T from ϕ to ϕ by

$$T(\wedge(q)) = \text{the truth value of a well-formed formula } [\wedge(q) \text{ and not } \wedge(q)]$$

for $q \subset \phi$, with $\wedge(q)$ denoting the conjunction of q.

We note that every subset q of ϕ becomes false by adding some well-formed formula f of ϕ. Hence, the conjunction of $q' = q \cup \{f\}$ is false and satisfies

$$T(\wedge(q')) = \wedge(q').$$

In this sense, ϕ is a fixed point of the map T.

Moreover, we have the followings.

1. In the sense that any subset q of ϕ is false if some well-formed formula is added to q, ϕ is **inconsistent**.

2. As ϕ is the class of all possible well-formed formulae, ϕ is **absolute**.

3. As ϕ is the totality of well-formed formulae, ϕ includes the well-formed formula whose meaning is that "ϕ is the class of all well-formed formulae in S" in some Gödel type correspondence between S and M_S. In this sense ϕ includes (the definition of) ϕ itself. Thus ϕ is **self-referential** and **self-creative**, and is **self-identical**, just as in M. C. Escher's lithograph in 1948, entitled "pencil drawing."

The item 3 implies that ϕ is a non-well founded set (see [48]).

The class ϕ is the first world, the Universe, which is completely chaotic. In other words, ϕ is "**absolute inconsistent self-identity**" in the sense of Kitarou Nishida [41], whose meaning was later clarified by Ronald Swan [46] in the form stated above. In this clarification, ϕ can be thought "absolute nothingness" in Hegel's sense.

The Universe ϕ is contradictory, and hence its truth value is constantly oscillating between the two extremal values or poles, truth and false, or $+1$ and -1, or more generally, inside a unit sphere of $\mathbf{C}$. Namely, the class ϕ as a set of wff's of the set theory S is countable, but the values which the elements of ϕ take vary on a unit sphere. In other words, the Universe ϕ is a stationary oscillation, when we see its meaning.

Oscillation is expressed by exponential functions: $\exp(ix \cdot p)$, where $x = (x_1, \cdots, x_d), p = (p_1, \cdots, p_d) \in R^d$ and $x \cdot p = \sum_{i=1}^d x_i p_i$, where d is a positive integer suggesting the dimension of space.

This $\exp(ix \cdot p)$ is an eigenfunction of the negative Laplacian $-\Delta$:

$$-\Delta = -\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}.$$

Namely

$$-\Delta \exp(ix \cdot p) = p^2 \exp(ix \cdot p).$$

This is generalized to some extent. I.e. if a perturbation $V = V(x)$ satisfies that

$$H = -\Delta + V(x) \text{ is a self-adjoint operator on } \mathcal{H} = L^2(R^d),$$

then

$$\phi \text{ is expressed as an eigenfunction of } H.$$

Considering the absolute nature of the Universe ϕ, we will be led to think that the Hamiltonian H of ϕ is a Hamiltonian of infinite degree of freedom on a Hilbert space:

$$\mathcal{U} = \{\phi\} = \bigoplus_{n=0}^{\infty} \left(\bigoplus_{\ell=0}^{\infty} \mathcal{H}^n \right) \quad (\mathcal{H}^n = \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n \text{ factors}}).$$

$\mathcal{U}$ is called a Hilbert space of possible universes. An element ϕ of $\mathcal{U}$ is called a universe and is of the form of an infinite matrix $(\phi_{n\ell})$ with components $\phi_{n\ell} \in \mathcal{H}^n$. $\phi = 0$ means $\phi_{n\ell} = 0$ for all n, ℓ.

Let $\mathcal{O} = \{S\}$ be the totality of the selfadjoint operators S in $\mathcal{U}$ of the form $S\phi = (S_{n\ell}\phi_{n\ell})$ for $\phi = (\phi_{n\ell}) \in \mathcal{D}(S) \subset \mathcal{U}$, where each component $S_{n\ell}$ is a selfadjoint operator in $\mathcal{H}^n$.

We now postulate that the Universe ϕ is an eigenfunction of the total Hamiltonian $H = H_{total}$.

Axiom 1. There is a selfadjoint operator $H_{total} = (H_{n\ell}) \in \mathcal{O}$ in $\mathcal{U}$ such that for some $\phi \in \mathcal{U} - \{0\}$ and $\lambda \in R^1$

$$H_{total}\phi \approx \lambda\phi \tag{11.1}$$

in the following sense: Let F_n be a finite subset of $\mathbf{N} = \{1, 2, \cdots\}$ with $\sharp(F_n)$(= the number of elements in F_n) $= n$ and let $\{F_n^\ell\}_{\ell=0}^\infty$ be the totality of such F_n (note: the set $\{F_n^\ell\}_{\ell=0}^\infty$ is countable). Then the formula (11.1) in the above means that there are integral sequences $\{n_k\}_{k=1}^\infty$ and $\{\ell_k\}_{k=1}^\infty$ and a real sequence $\{\lambda_{n_k\ell_k}\}_{k=1}^\infty$ such that $F_{n_k}^{\ell_k} \subset F_{n_{k+1}}^{\ell_{k+1}}$; $\bigcup_{k=1}^\infty F_{n_k}^{\ell_k} = \mathbf{N}$;

$$H_{n_k\ell_k}\phi_{n_k\ell_k} = \lambda_{n_k\ell_k}\phi_{n_k\ell_k}, \quad \phi_{n_k\ell_k} \neq 0, \quad k = 1, 2, 3, \cdots ; \tag{11.2}$$

and

$$\lambda_{n_k\ell_k} \to \lambda \quad \text{as} \quad k \to \infty.$$

Here we should repeat a remark in Axiom 2.2: the subscript ℓ in $H_{n\ell}$ distinguishes different local systems with the same number $N = n + 1$ of particles.

$H = H_{total}$ is an infinite matrix $(H_{n\ell})$ of selfadjoint operators $H_{n\ell}$ in $\mathcal{H}^n$. Axiom 1 asserts that this matrix converges in the sense of (11.1) on our universe ϕ. We remark that our universe ϕ is not determined uniquely by this condition.

The universe as a state ϕ is a whole, within which is all. As such a whole, the state ϕ can follow the two ways: The one is that ϕ develops along a global time T in the grand universe $\mathcal{U}$ under a propagation $\exp(-iTH_{total})$, and another is that ϕ is a bound state of H_{total}. If there were such a global time T as in the first case, all phenomena had to develop along that global time T, and the locality of time would be lost. We could then *not* construct a notion of local times compatible with general theory of relativity. The only one possibility is therefore to adopt the stationary universe ϕ of Axiom 1.

In every finite part of ϕ, a local existence in ϕ is expressed by a superposition of exponential functions

$$\psi(x) = (2\pi)^{-d(N-1)/2} \int_{R^{d(N-1)}} \exp(ix \cdot p)g(p)dp$$

for some natural number $N = n + 1 \geq 2$ with n corresponding to the superscript n in $\mathcal{H}^n$ of the definition of $\mathcal{U}$ above. The function $g(p)$ is called Fourier transform of $\psi(x)$ and satisfies

$$g(p) = \mathcal{F}\psi(p) := (2\pi)^{-d(N-1)/2} \int_{R^{d(N-1)}} \exp(-ip \cdot y)\psi(y)dy.$$

A finite subset of wff's in ϕ corresponds to a partial Hamiltonian H of H_{total} of finite degree of freedom, as the content/freedom that is given by a finite number of wff's in ϕ corresponds to a finite degree, $n = N - 1$, of freedom of a partial wave function $\psi(x)$ of the total wave function ϕ. If such a partial Hamiltonian H of H_{total} satisfies some conditions, we can get a similar expansion of a local existence $\psi(x)$ by using generalized eigenfunctions of H. This is known as a spectral representation of H in a general setting, but we here are speaking of a more specific expression called generalized Fourier transform or generalized eigenfunction expansion associated with Hamiltonian H (originated by Teruo Ikebe [16]).

We call p momentum conjugate to x. More precisely we define momentum operator $P = (P_1, \cdots, P_d)$ conjugate to configuration operator $X = (X_1, \cdots, X_d)$ (X_j = multiplication operator by configuration x_j) by

$$P_j = \mathcal{F}^{-1} p_j \mathcal{F} = \frac{1}{i}\frac{\partial}{\partial x_j} \qquad (j = 1, \cdots, d).$$

Then P and X satisfy

$$[P_j, X_\ell] = P_j X_\ell - X_\ell P_j = \delta_{j\ell}\frac{1}{i}.$$

This shows that what we are dealing with is quantum mechanics. So to accord with actual observation, we modify the definition of P

$$P_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j},$$

where $\hbar = h/(2\pi)$, and h is Planck constant. Accordingly, the Fourier and inverse Fourier transformations are modified

$$\mathcal{F}\psi(p) = g(p) = (2\pi\hbar)^{-d(N-1)/2} \int_{R^{d(N-1)}} \exp(-ip \cdot y/\hbar)\psi(y)dy,$$
$$\mathcal{F}^{-1}g(x) = (2\pi\hbar)^{-d(N-1)/2} \int_{R^{d(N-1)}} \exp(ix \cdot p/\hbar)g(p)dp.$$

To sum our arguments up to here, we have constructed quantum mechanics as a semantics of the class ϕ of all well-formed formulae of a formal set theory S. Quantum mechanics is, in this context, given as an interpretation of set theory.

We continue to complete our semantics of the Universe ϕ.

A local existence is of finite nature, and it is so local that it cannot know the existence of the infinite Universe, and is self-centered. In other words, a local coordinates system starts from its own origin, and it is the self-centered origin of the local system. All things are measured with respect to this local origin.

Therefore we have our second and third principles.

Axiom 2. (a simplified version of Axiom 2.1) A local system is of finite nature, having its own origin of position X and momentum P, independent of others' origins and others' inside worlds.

Axiom 3. (a simplified version of Axiom 2.2) The nature of locality is expressed by a local Hamiltonian

$$H = -\frac{1}{2}\Delta + V$$

up to some perturbation V, that does not violate the oscillatory nature of local existence. Here $\Delta = \sum_{j=1}^{N-1} \frac{\hbar^2}{\mu_j} \sum_{k=1}^{d} \frac{\partial^2}{\partial x_{jk}^2}$, the number N corresponds to the number of quantum particles of the local system, and μ_j is the reduced masses of the particles of the local system.

A local existence (or local system) is oscillating as a sum or integral of generalized eigenfunctions of H. In this sense, the locality or local system is a *stationary oscillating system.*

A local oscillation may be an eigenfunction of the local Hamiltonian H. However, by the very nature that locality is a self-centered existence of finite nature, it is shown that it cannot be an eigenstate of H, or more precisely speaking, there is at least one Universe wave function ϕ every part of which is not an eigenfunction of the local system Hamiltonian H. (See chapter 12. See also [28], [29], [26], [31], [30].)

To express this oscillation explicitly in some "outer coordinate," we force the locality or local system to oscillate along an "afterward-introduced" real-valued parameter t. The oscillation is then expressed by using the Hamiltonian H

$$\exp(-2\pi i t H/h).$$

This operator is known in quantum mechanics as the evolution operator of the local system. We will call it the local clock of the system, and we will call t the local time of the system.

Using our self-centered coordinates of our local system in Axiom 2, that is, letting x be position coordinates and $v = m^{-1}P$ be velocity coordinates inside the local system (m being some diagonal mass matrix), we can prove, by virtue of the fact that a local oscillation $\psi(x)$ is not an eigenfunction of H, that

$$\left(\frac{x}{t} - v\right) \exp(-itH/\hbar)\psi(x) \to 0$$

as t tends to $\pm\infty$ along some sequence in some spectral decomposition of $\exp(-itH/\hbar)\psi$ (Theorem 3.2 or see [26]). This means that the word "local clock" is appropriate for the operator $\exp(-itH/\hbar)$ and so is "local time" for the parameter t. Therefore we also have seen that "time" exists locally and only locally, exactly by the fact that locality is a self-centered existence of finite nature. This fact corresponds to Ronald Swan's statements in page 27 of [46] "localization must be completely, or unconditionally, circumstantial" and "localization is not self-creative."

Let P_H denote the orthogonal projection onto the space of bound states for a self-adjoint operator H. We call the set of all states orthogonal to the space of bound states a scattering space, and its element as a scattering state. Let $\phi = (\phi_{n\ell})$ with $\phi_{n\ell} = \phi_{n\ell}(x_1, \cdots, x_n) \in L^2(R^{dn})$ be the universe in Axiom 1, and let $\{n_k\}$ and $\{\ell_k\}$ be the sequences specified there. Let $x^{(n,\ell)}$ denote the relative coordinates of $n+1$ particles in F^{ℓ}_{n+1}.

Definition 1.

(1) We define $\mathcal{H}_{n\ell}$ as the sub-Hilbert space of $\mathcal{H}^n$ generated by the functions $\phi_{n_k \ell_k}(x^{(n,\ell)}, y)$ of $x^{(n,\ell)} \in R^{dn}$ with regarding $y \in R^{d(n_k - n)}$ as a parameter, where k moves over a set $\{k \mid n_k \geq n, F^{\ell}_{n+1} \subset F^{\ell_k}_{n_k+1}, k \in \mathbf{N}\}$.

(2) $\mathcal{H}_{n\ell}$ is called a *local universe* of ϕ.

(3) $\mathcal{H}_{n\ell}$ is said to be non-trivial if $(I - P_{H_{n\ell}})\mathcal{H}_{n\ell} \neq \{0\}$.

The total universe ϕ is a single element in $\mathcal{U}$. The local universe $\mathcal{H}_{n\ell}$ can be richer and may have elements more than one. This is because we consider the subsystems of the universe consisting of a finite number of particles. These subsystems receive the influence from the other particles of infinite number outside the subsystems, and may vary to constitute a non-trivial subspace $\mathcal{H}_{n\ell}$. We will consider this point in chapter 12.

We can now define local system.

Definition 2.

(1) The restriction of H_{total} to $\mathcal{H}_{n\ell}$ is also denoted by the same notation $H_{n\ell}$ as the (n, ℓ)-th component of H_{total}.

(2) We call the pair $(H_{n\ell}, \mathcal{H}_{n\ell})$ a local system.

(3) The unitary group $e^{-itH_{n\ell}}$ $(t \in R^1)$ on $\mathcal{H}_{n\ell}$ is called the *local* or *proper clock* of the local system $(H_{n\ell}, \mathcal{H}_{n\ell})$, if $\mathcal{H}_{n\ell}$ is non-trivial: $(I - P_{H_{n\ell}})\mathcal{H}_{n\ell} \neq \{0\}$. (Note that the clock is defined only for $N = n + 1 \geq 2$, since $H_{0\ell} = 0$ and $P_{H_{0\ell}} = I$.)

(4) The universe ϕ is called *rich* if $\mathcal{H}_{n\ell}$ equals $\mathcal{H}^n = L^2(R^{dn})$ for all $n \geq 1$, $\ell \geq 0$. For a rich universe ϕ, $H_{n\ell}$ equals the (n, ℓ)-th component of H_{total}.

Definition 3.

(1) The parameter t in the exponent of the local clock $e^{-itH_{n\ell}} = e^{-itH_{(N-1)\ell}}$ of a local system $(H_{n\ell}, \mathcal{H}_{n\ell})$ is called the (quantum-mechanical) *proper time* or *local time* of the local system $(H_{n\ell}, \mathcal{H}_{n\ell})$, if $(I - P_{H_{n\ell}})\mathcal{H}_{n\ell} \neq \{0\}$.

(2) This time t is denoted by $t_{(H_{n\ell}, \mathcal{H}_{n\ell})}$ indicating the local system under consideration.

This definition is a one reverse to the usual definition of the motion or dynamics of the N-body quantum systems, where the time t is given *a priori* and then the motion of the particles is defined by $e^{-itH_{(N-1)\ell}}f$ for a given initial state f of the system.

Time is thus defined only for local systems $(H_{n\ell}, \mathcal{H}_{n\ell})$ and is determined by the associated local clock $e^{-itH_{n\ell}}$. Therefore there are infinitely many number of times $t = t_{(H_{n\ell}, \mathcal{H}_{n\ell})}$ each of which is proper to the local system $(H_{n\ell}, \mathcal{H}_{n\ell})$. In this sense time is a local notion. There is no time for the total universe ϕ in Axiom 1, which is a bound state of the total Hamiltonian H_{total} in the sense of the condition (11.1) of Axiom 1.

Once given the local time, the local system obeys Schrödinger equation

$$\left(\frac{\hbar}{i}\frac{d}{dt} + H\right) \exp(-itH/\hbar)\psi(x) = 0.$$

All up to now can be expressed on a Euclidean space R^d. We need not worry about any curvature as we consider ourselves with respect to our own coordinates.

But when we look at the outside world, our view will be distorted due to the finiteness of our ability. As equivalent existences as localities, we are all subject to one and the same law of distortion.

Among local systems, we thus pose a law of democracy.

Axiom 4. (a simplified version of Axiom 8.1) General Principle of Relativity. Physical worlds or laws are the same for all local observers.

As a locality, we cannot distinguish between the actual force and the fictitious force, as far as the force is caused by the distortions that our confrontations to the outside world produce.

We thus have the fifth axiom.

Axiom 5. (a simplified version of Axiom 8.2) Principle of Equivalence. For any gravitational force, we can choose a coordinate system (as a function of time t) where the effect of gravitation vanishes.

Axioms 4 and 5 are concerned with the distortion of our view when we meet the outside, while Axioms 1–3 are about the inside world which is independently conceived as its own. The oscillatory nature of local systems in Axiom 3 is a consequence of the locality of the system and the stationary nature of the Universe, so that the oscillation is due to the intrinsic "internal" cause, while the distortion of our view to the outside is due to observational "external" cause.

Those two aspects, the internal and the external aspects, are independent mutually, because the internal coordinate system of a local system is a relative one inside the local system and does not have any relation with the external coordinates. Therefore, when we are inside, we are free from the distortion, while when we are meeting the outside, we are in a state that we forget the inside and see a curved world. Thus Axioms 1–5 are consistent.

Quantum mechanics is introduced as a semantic interpretation of a formal set theory, and general relativity is set as a democracy principle among finite, local systems. The origin of local time is in this finitude of local existence, and it gives the general relativistic proper time of each system.

Set theory is a purely inward thought. Physics obtained as semantics of the set theory is a look at it from the outside. The obtained QM itself is a description of the inside world that breeds set theory. The self-reference prevails everywhere and at every stage.

Chapter 12

Existence of Local Motion

We are in a position to see how the stationary nature of the universe and the existence of local motion and hence local time are compatibly incorporated into our formulation.

12.1 Gödel's theorem

Our starting point is the incompleteness theorem proved by Gödel [15]. It states that any consistent formal theory that can describe number theory includes an infinite number of undecidable propositions (see chapter 10). The physical world includes at least natural numbers, and it is described by a system of words, which can be translated into a formal physics theory. The theory of physics, if consistent, therefore includes an undecidable proposition, i.e. a proposition whose correctness cannot be known by human beings until one finds a phenomenon or observation that supports the proposition or denies the proposition. Such propositions exist infinitely according to Gödel's theorem. Thus human beings, or any other finite entity, will never be able to reach a "final" theory that can express the totality of the phenomena in the universe.

Thus we have to assume that any human observer sees a part or subsystem L of the universe and never gets the total Hamiltonian H_{total} in (11.1) by his observation. Here the total Hamiltonian H_{total} is an *ideal* Hamiltonian that might be gotten by "God." In other words, a consequence from Gödel's theorem is that the Hamiltonian that an observer assumes with his observable universe is a part H_L of H_{total}. Stating explicitly, the consequence from Gödel's theorem is the following proposition

$$H_{total} = H_L + I + H_E, \quad H_E \neq 0, \tag{12.1}$$

where H_E is an unknown Hamiltonian describing the system E exterior to the realm of the observer, whose existence, i.e. $H_E \neq 0$, is assured by Gödel's theorem. This unknown system E includes all that is unknown to the observer. E.g., it might contain particles which exist near us but have not been discovered yet, or are unobservable for some reason at the time of observation. The term I is an unknown interaction between the observed system L and the unknown system E. Since the exterior system E is assured to exist by Gödel's theorem, the interaction I does not vanish: In fact assume I vanishes. Then the observed system L and the exterior system E do not interact, which is the same as that the exterior system E does not exist for the observer. On the other hand, assigning the so-called Gödel number to each proposition in number theory, Gödel constructs undecidable propositions in number theory by a diagonal argument, which shows that any consistent formal theory has a region exterior to the knowable world (see [15]). Thus the observer must be able to construct a proposition by Gödel's procedure that proves E exists, which means $I \neq 0$. By the same reason, I is not a constant operator:

$$I \neq \text{constant operator.} \tag{12.2}$$

For suppose it is a constant operator. Then the systems L and E do not change no matter how far or how near they are located because the interaction between L and E is a constant operator. This is the same situation as that the interaction does not exist, thus reduces to the case $I = 0$ above.

We now arrive at the following observation: For an observer, the observable universe is a part L of the total universe and it looks as though it follows the Hamiltonian H_L, not following the total Hamiltonian H_{total}. And the state of the system L is described by a part $\phi(\cdot, y)$ of the state ϕ of the total universe, where y is an unknown coordinate of system L inside the total universe, and $\cdot$ is the variable controllable by the observer, which we will denote by x.

12.2 Local time exists

In the following argument, we assume an exact relation:

$$H_{total}\phi = 0 \tag{12.3}$$

instead of (11.1), for simplicity.

Assume now, as is usually expected under condition (12.3), that there is no local time of L, i.e. that the state $\phi(x, y)$ is an eigenstate of the local Hamiltonian H_L for some $y = y_0$ and a real number μ:

$$H_L\phi(x, y_0) = \mu\phi(x, y_0). \tag{12.4}$$

Then from (12.1), (12.3) and (12.4) follows that

$$\begin{aligned} 0 = H_{total}\phi(x, y_0) &= H_L\phi(x, y_0) + I(x, y_0)\phi(x, y_0) + H_E\phi(x, y_0) \\ &= (\mu + I(x, y_0))\phi(x, y_0) + H_E\phi(x, y_0). \end{aligned} \tag{12.5}$$

Here x varies over the possible positions of the particles inside L. On the other hand, since H_E is the Hamiltonian describing the system E exterior to L, it does not affect the variable x and acts only on the variable y. Thus $H_E\phi(x, y_0)$ varies as a bare function $\phi(x, y_0)$ insofar as the variable x is concerned. Equation (12.5) is now written: For all x

$$H_E\phi(x, y_0) = -(\mu + I(x, y_0))\phi(x, y_0). \tag{12.6}$$

As we have seen in (12.2), the interaction I is not a constant operator and varies when x varies[6], whereas the action of H_E on ϕ does not. Thus there is a nonempty set of points x_0 where $H_E\phi(x_0, y_0)$ and $-(\mu + I(x_0, y_0))\phi(x_0, y_0)$ are different, and (12.6) does not hold at such points x_0. If I is assumed to be continuous in the variables x and y, these points x_0 constitutes a set of positive measure. This then implies that our assumption (12.4) is wrong. Thus a subsystem L of the universe cannot be a bound state with respect to the observer's Hamiltonian H_L. This means that the system L is observed as a non-stationary system, therefore there must be observed a motion inside the system L. This proves that the "time" of the local system L *exists for the observer* as a measure of motion, whereas the total universe is stationary and does not have "time."

12.3 A refined argument

To show the argument in section 12.2 more explicitly, we consider a simple case of

$$H_{total} = \frac{1}{2}\sum_{k=1}^{N} h^{ab}(X_k)p_{ka}p_{kb} + V(X).$$

Here N $(1 \le N \le \infty)$ is the number of particles in the universe, h^{ab} is a three-metric, $X_k \in R^d$ is the position of the k-th particle, p_{ka} is a functional derivative corresponding to momenta of the k-th particle, and $V(X)$ is a potential. The configuration $X = (X_1, X_2, \cdots, X_N)$ of total particles is decomposed as $X = (x, y)$ accordingly to if the k-th particle is inside L or not, i.e. if the k-th particle is in L, X_k is a component of x and if not it is that of y. H_{total} is decomposed as follows:

$$H_{total} = H_L + I + H_E.$$

Here H_L is the Hamiltonian of a subsystem L that acts only on x, H_E is the Hamiltonian describing the exterior E of L that acts only on y, and $I = I(x, y)$ is the interaction between the systems L and E. Note that H_L and H_E commute.

Theorem 12.1 *Let P denote the eigenprojection onto the space of all bound states of H_{total}. Let P_L be the eigenprojection for H_L. Then we have*

$$(1 - P_L)P \neq 0, \tag{12.7}$$

unless the interaction $I = I(x, y)$ is a constant with respect to x for any y.

Proof: Assume that (12.7) is incorrect. Then we have

$$P_L P = P.$$

[6]Note that Gödel's theorem applies to any fixed $y = y_0$ in (12.2). Namely, for any position y_0 of the system L in the universe, the observer must be able to know that the exterior system E exists because Gödel's theorem is a universal statement valid throughout the universe. Hence $I(x, y_0)$ is not a constant operator with respect to x for any fixed y_0.

Taking the adjoint operators on the both sides, we then have

$$PP_L = P.$$

Thus $[P_L, P] = P_L P - PP_L = 0$. But in generic this does not hold because

$$[H_L, H_{total}] = [H_L, H_L + I + H_E] = [H_L, I] \neq 0,$$

unless $I(x, y)$ is equal to a constant with respect to x. Q.E.D.

Remark. In the context of chapter 11, the theorem implies the following:

$$(1 - P_L)P\mathcal{U} \neq \{0\},$$

where $\mathcal{U}$ is a Hilbert space consisting of all possible states ϕ of the total universe. This relation implies that there is a vector $\phi \neq 0$ in $\mathcal{U}$ which satisfies $H_{total}\phi = \lambda\phi$ for a real number λ while $H_L\Phi \neq \mu\Phi$ for any real number μ, where $\Phi = \phi(\cdot, y)$ is a state vector of the subsystem L with an appropriate choice of the position y of the subsystem. Thus the space generated by $\phi(\cdot, y)$'s when y varies is non-trivial in the sense of Definition 1 in chapter 11, which proves for the universe ϕ that any local system L is non-trivial, and hence proves the existence of local time for any local system of the universe ϕ. Thus we have at least one stationary universe ϕ where every local system has its local time.

Exercise

We consider the systems $S^{(\alpha)}$ defined in chapter 10.

Let $\tilde{q}(\alpha)$ denote the Gödel number of Rosser formula or its negation $A_{(\alpha)}$ $(= A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$ or $\neg A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)}))$, if the Rosser formula $A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$ is well-defined.

By "recursive ordinals" we mean those defined by Rogers [42]. Then that α is a recursive ordinal means that $\alpha < \omega_1^{CK}$, where ω_1^{CK} is the so-called Church-Kleene ordinal ([12], [47]).

Lemma. The number $\tilde{q}(\alpha)$ is recursively defined for countable recursive ordinals $\alpha < \omega_1^{CK}$. Here 'recursively defined' means that $\tilde{q}(\alpha)$ is defined inductively starting from 0.

Remark. The original meaning of 'recursive' is 'inductive.' The meaning of the word 'recursive' in the following is the one that matches the spirit of Kleene [33] (especially, the spirit of the inductive construction of metamathematical predicates described in section 51 of [33]).

Proof. The well-definedness of $\tilde{q}(0)$ is assured by Rosser-Gödel theorem as explained in chapter 10.

We make an induction hypothesis that for each $\delta < \alpha$, the Gödel number $\tilde{q}(\gamma)$ of the formula $A_{(\gamma)}$ $(= A_{q^{(\gamma)}}(\mathbf{q}^{(\gamma)})$ or $\neg A_{q^{(\gamma)}}(\mathbf{q}^{(\gamma)}))$ with $\gamma \le \delta$ is recursively defined for $\gamma \le \delta$.

We want to prove that the Gödel number $\tilde{q}(\gamma)$ is recursively well-defined for $\gamma \le \alpha$.

i) When $\alpha = \delta + 1$, by induction hypothesis we can determine recursively whether or not a given formula A_r with Gödel number r is equal to one of the axiom formulas $A_{(\gamma)}$ $(\gamma \le \delta)$ of $S^{(\alpha)}$. In fact, we have only to see, for a finite number of γ's with $\tilde{q}(\gamma) \le r$ and $\gamma \le \delta$, if we have $A_{(\gamma)} = A_r$ or not. By induction hypothesis that $\tilde{q}(\gamma)$ is recursively well-defined for $\gamma \le \delta$, this is then decided recursively.

Thus Gödel predicate $\mathbf{A}^{(\alpha)}(a, b)$ and Rosser predicate $\mathbf{B}^{(\alpha)}(a, c)$ with superscript α are recursively defined, and hence are numeralwise expressible in $S^{(\alpha)}$. Then the Rosser formula $A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$ is well-defined, and the Gödel number $\tilde{q}(\alpha)$ of Rosser formula or its negation $A_{(\alpha)}$ $(= A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$ or $\neg A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)}))$ is defined recursively. Thus $\tilde{q}(\gamma)$ is recursively well-defined for $\gamma \le \alpha$.

ii) If α is a countable *recursive* limit ordinal, then there is an increasing sequence of recursive ordinals $\alpha_n < \alpha$ such that

$$\alpha = \bigcup_{n=0}^{\infty} \alpha_n. \tag{12.8}$$

In the system $S^{(\alpha)}$, the totality of the added axioms $A_{(\gamma)}$ $(\gamma < \alpha)$ is the sum of the added axioms $A_{(\gamma)}$ $(\gamma < \alpha_n)$ of $S^{(\alpha_n)}$. By induction hypothesis, $\tilde{q}(\gamma)$ is recursively defined for $\gamma < \alpha_n$. Thus in each $S^{(\alpha_n)}$ we can determine recursively whether or not a given formula A_r is an axiom of $S^{(\alpha_n)}$ by seeing, for a finite number of γ's with $\tilde{q}(\gamma) \le r$ and $\gamma < \alpha_n$, if $A_{(\gamma)} = A_r$ or not.

This is extended to $S^{(\alpha)}$. To see this, we have only to see the γ's with $\tilde{q}(\gamma) \le r$ and $\gamma < \alpha$, and determine for those finite number of γ's if $A_{(\gamma)} = A_r$ or not. By (12.8),

$$\tilde{q}(\gamma) \le r \ \text{ and } \ \gamma < \alpha \Leftrightarrow \exists n \ \text{ such that } \tilde{q}(\gamma) \le r \ \text{ and } \ \gamma < \alpha_n.$$

Then by induction on n with using the result in the above paragraph for $S^{(\alpha_n)}$ and noting that the bound r on $\tilde{q}(\gamma)$ is uniform in n, we can show that the condition whether or not $\tilde{q}(\gamma) \leq r$ and $\gamma < \alpha$ is recursively determined. Whence the question whether or not a given formula A_r is one of the axioms $A_{(\gamma)}$ of $S^{(\alpha)}$ with $\tilde{q}(\gamma) \leq r$ and $\gamma < \alpha$ is determined recursively. Thus Gödel predicate $\mathbf{A}^{(\alpha)}(a,b)$ and Rosser predicate $\mathbf{B}^{(\alpha)}(a,c)$ with superscript α are recursively defined, and hence are numeralwise expressible in $S^{(\alpha)}$. Therefore the Rosser formula $A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$ is well-defined, and the Gödel number $\tilde{q}(\alpha)$ of Rosser formula or its negation $A_{(\alpha)}$ $(= A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$ or $\neg A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)}))$ is defined recursively. Thus $\tilde{q}(\gamma)$ is recursively well-defined for $\gamma \leq \alpha$. This completes the proof of the lemma.

Assume now that α is a countable limit ordinal such that there is an increasing sequence of recursive ordinals $\alpha_n < \alpha$ with

$$\alpha = \bigcup_{n=0}^{\infty} \alpha_n. \tag{12.9}$$

An actual example of such an α is the Church-Kleene ordinal ω_1^{CK}.

In the system $S^{(\alpha)}$, the totality of the added axioms $A_{(\gamma)}$ $(\gamma < \alpha)$ is the sum of the added axioms $A_{(\gamma)}$ $(\gamma < \alpha_n)$ of $S^{(\alpha_n)}$. By the lemma, $\tilde{q}(\gamma)$ is recursively defined for $\gamma < \alpha_n$. Thus in each $S^{(\alpha_n)}$ we can determine recursively whether or not a given formula A_r is an axiom of $S^{(\alpha_n)}$ by seeing, for a finite number of γ's with $\tilde{q}(\gamma) \leq r$ and $\gamma < \alpha_n$, if $A_{(\gamma)} = A_r$ or not.

This is extended to $S^{(\alpha)}$. To see this, we have only to see the γ's with $\tilde{q}(\gamma) \leq r$ and $\gamma < \alpha$, and determine for those finite number of γ's if $A_{(\gamma)} = A_r$ or not. By (12.9),

$$\tilde{q}(\gamma) \leq r \ \text{ and } \ \gamma < \alpha \Leftrightarrow \exists n \ \text{ such that } \tilde{q}(\gamma) \leq r \ \text{ and } \ \gamma < \alpha_n.$$

Then by induction on n with using the above result for $S^{(\alpha_n)}$ in the preceding paragraph and noting that the bound r on $\tilde{q}(\gamma)$ is uniform in n, we can show that the condition whether or not $\tilde{q}(\gamma) \leq r$ and $\gamma < \alpha$ is recursively determined. Then within those finite number of γ's with $\tilde{q}(\gamma) \leq r$ and $\gamma < \alpha$, we can decide recursively if for some $\gamma < \alpha$ with $\tilde{q}(\gamma) \leq r$, we have $A_r = A_{(\gamma)}$ or not. Therefore we can determine recursively whether or not a given formula A_r is an axiom of $S^{(\alpha)}$.

Therefore Gödel predicate $\mathbf{A}^{(\alpha)}(a,b)$ and Rosser predicate $\mathbf{B}^{(\alpha)}(a,c)$ are recursively defined, and hence are numeralwise expressible in $S^{(\alpha)}$. Then the Gödel number $q^{(\alpha)}$ of the formula

$$\forall b[\neg A^{(\alpha)}(a,b) \vee \exists c(c \leq b \,\&\, B^{(\alpha)}(a,c))]$$

is well-defined, and hence Rosser formula $A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$ is well-defined and Rosser-Gödel theorem applies to the system $S^{(\alpha)}$. Therefore we can extend $S^{(\alpha)}$ consistently by adding one of Rosser formula or its negation $A_{(\alpha)}$ $(= A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$ or $\neg A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)}))$ to the axioms of $S^{(\alpha)}$ and get a consistent system $S^{(\alpha+1)}$.

In particular if we assume a least nonrecursive ordinal ω_1^{CK} exists and take $\alpha = \omega_1^{CK}$, we get a consistent system $S^{(\omega_1^{CK}+1)}$. This contradicts the case ii) of theorem 10.6 in chapter 10. We leave the following problem to the reader.

Question. The least nonrecursive ordinal, the so-called Church-Kleene ordinal ω_1^{CK} has been assumed to give a bound on recursive construction of formal systems (see [12], [44], [47]). However the above argument seems to question if ω_1^{CK} really exists in usual set theoretic sense. How should we think?

Bibliography

[1] R. Abraham and J. E. Marsden, Foundations of Mechanics, The Benjamin/Cummings Publishing Company, 2nd ed., London-Amsterdam-Don Mills, Ontario-Sydney-Tokyo, 1978.

[2] A. Ashtekar and J. Stachel (eds.), *Conceptual Problems of Quantum Gravity*, Birkhäuser, Boston-Basel-Berlin, 1991.

[3] Noted by Dr. Peter Beamish at Ceta Research (http://www.oceancontact.com/) in a post http://groups.yahoo.com/group/time/message/2054 to the Time mailing list at http://groups.yahoo.com/group/time/.

[4] P. Busch and P. J. Lahti, *The determination of the past and the future of a physical system in quantum mechanics*, Foundations of Physics, **19** (1989), 633-678.

[5] H. L. Cycon *et al.*, Schrödinger Operators, Springer-Verlag, 1987.

[6] J. Dereziński, *Asymptotic completeness of long-range N-body quantum systems*, Annals of Math. **138** (1993), 427-476.

[7] A. Einstein, *On the electrodynamics of moving bodies*, translated by W. Perrett and G. B. Jeffery from *Zur Elektrodynamik bewegter Körper*, Annalen der Physik, **17** (1905) in The Principle of Relativity, Dover, 1952.

[8] V. Enss, *Asymptotic completeness for quantum mechanical potential scattering I. Short-range potentials*, Commun. Math. Phys., **61** (1978), 285-291.

[9] V. Enss, *Quantum scattering theory for two- and three-body systems with potentials of short and long range*, in "Schrödinger Operators" edited by S. Graffi, Springer Lecture Notes in Math. **1159**, Berlin 1985, 39-176.

[10] V. Enss, *Introduction to asymptotic observables for multiparticle quantum scattering*, in "Schrödinger Operators, Aarhus 1985" edited by E. Balslev, Lect. Note in Math., vol. 1218, Springer-Verlag, 1986, 61-92.

[11] V. Enss, *Long-range scattering of two- and three-body quantum systems*, Equations aux derivées partielles, Publ. Ecole Polytechnique, Palaiseau (1989), 1-31.

[12] S. Feferman, *Transfinite recursive progressions of axiomatic theories*, Journal Symbolic Logic, **27** (1962), 259-316.

[13] R. Froese and I. Herbst, *Exponential bounds and absence of positive eigenvalues for N-body Schrödinger operators*, Commun. Math. Phys. **87** (1982), 429-447.

[14] D. Gilberg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[15] K. Gödel, *On formally undecidable propositions of Principia mathematica and related systems I*, in "Kurt Gödel Collected Works, Volume I, Publications 1929-1936," Oxford University Press, New York, Clarendon Press, Oxford, 1986, 144-195, translated from *Über formal unentsceidebare Sätze der Principia mathematica und verwandter Systeme I*, Monatshefte für Mathematik und Physik, **38** (1931), 173-198.

[16] T. Ikebe, *Eigenfunction expansions associated with the Schrödinger operators and their applications to scattering theory*, Arch. Rational Mech. Anal., **5** (1960), 1-34.

[17] M. Insall, private communication, 2003, (an outline is found at: http://www.cs.nyu.edu/pipermail/fom/2003-June/006862.html).

[18] C. J. Isham, *Canonical quantum gravity and the problem of time*, Proceedings of the NATO Advanced Study Institute, Salamanca, June 1992, Kluwer Academic Publishers, 1993.

[19] H. Isozaki and H. Kitada, *Modified wave operators with time-independent modifiers*, Journal of the Fac. Sci, University of Tokyo, Sec. IA, **32** (1985), 77-104.

[20] H. Isozaki and H. Kitada, *Scattering matrices for two-body Schrödinger operators*, Scientific Papers of the College of Arts and Sciences, The University of Tokyo **35** (1986), 81-107.

[21] T. Kato, Perturbation Theory for Linear operators, Springer-Verlag, 1976.

[22] T. Kato and S. T. Kuroda, *Theory of simple scattering and eigenfunction expansions*, Functional Analysis and Related Fields, Springer-Verlag, Berlin, Heidelberg, and New York, 1970, 99-131.

[23] H. Kitada, *Fourier integral operators with weighted symbols and micro-local resolvent estimates*, J. Math. Soc. Japan, **39** (1987), 101-124.

[24] H. Kitada, *Asymptotic completeness of N-body wave operators I. Short-range quantum systems*, Rev. in Math. Phys. **3** (1991), 101-124.

[25] H. Kitada, *Asymptotic completeness of N-body wave operators II. A new proof for the short-range case and the asymptotic clustering for long-range systems*, Functional Analysis and Related Topics, 1991, Ed. by H. Komatsu, Lect. Note in Math., vol. 1540, Springer-Verlag, 1993, 149-189.

[26] H. Kitada, *Theory of local times*, Il Nuovo Cimento **109 B, N. 3** (1994), 281-302, http://xxx.lanl.gov/abs/astro-ph/9309051.

[27] H. Kitada, *Quantum Mechanics and Relativity – Their Unification by Local Time*, Spectral and Scattering Theory, edited by A. G. Ramm, Plenum Publishers, New York 1998, 39-66. (http://xxx.lanl.gov/abs/gr-qc/9612043)

[28] H. Kitada, *A possible solution for the non-existence of time*, http://xxx.lanl.gov/abs/gr-qc/9910081, 1999.

[29] H. Kitada, *Local Time and the Unification of Physics Part II. Local System*, http://xxx.lanl.gov/abs/gr-qc/0110066, 2001.

[30] H. Kitada and L. Fletcher, *Local time and the unification of physics, Part I: Local time*, Apeiron **3** (1996), 38-45.

[31] H. Kitada and L. Fletcher, *Comments on the Problem of Time*, http://xxx.lanl.gov/abs/gr-qc/9708055, 1997.

[32] H. Kitada and H. Kumano-go, *A family of Fourier integral operators and the fundamental solution for a Schrödinger equation*, Osaka J. Math., **18** (1981), 291-360.

[33] S. C. Kleene, Introduction to Metamathematics, North-Holland Publishing Co. Amsterdam, P. Noordhoff N. V., Groningen, 1964.

[34] H. Kumano-go, Pseudo-differential Operators, Iwanami-Shoten, 1974 (In Japanese).

[35] E. H. Lieb, *The stability of matter: From atoms to stars*, Bull. Amer. Math. Soc. **22** (1990), 1-49.

[36] C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation, W. H. Freeman and Company, New York, 1973.

[37] T. S. Natarajan, private communication, 2000.

[38] T. S. Natarajan, Phys. Essys., **9, No. 2** (1996) 301-310, or *Do Quantum Particles have a Structure?* (http://www.geocities.com/ResearchTriangle/Thinktank/1701/).

[39] J. von Neumann, "Mathematical Foundations of Quantum Mechanics," translated by R. T. Beyer, Princeton University Press, Princeton, New Jersey, 1955.

[40] I. Newton, Sir Isaac Newton Principia, Vol. I The Motion of Bodies, Motte's translation Revised by Cajori, Tr. Andrew Motte ed. Florian Cajori, Univ. of California Press, Berkeley, Los Angeles, London, 1962.

[41] Kitarou Nishida, *Absolute inconsistent self-identity* (*Zettai-Mujunteki-Jikodouitsu*), http://www.aozora.gr.jp/cards/000182/files/1755.html, 1989.

[42] H. Rogers Jr., Theory of Recursive Functions and Effective computability, McGraw-Hill, 1967.

[43] L. I. Schiff, Quantum Mechanics, McGRAW-HILL, New York, 1968.

[44] U. R. Schmerl, *Iterated reflection principles and the ω-rule*, Journal Symbolic Logic, **47** (1982), 721–733.

[45] I. M. Sigal and A. Soffer, *The N-particle scattering problem: Asymptotic completeness for short-range systems*, Ann. Math. **126** (1987), 35-108.

[46] Ronald Swan, *A meta-scientific theory of nature and the axiom of pure possibility*, a draft not for publication, 2002.

[47] A. M. Turing, *Systems of logic based on ordinals*, Proc. London Math. Soc., ser. 2, **45** (1939), 161–228.

[48] P. Wegner and D. Goldin, *Mathematical models of interactive computing*, draft, January 1999, Brown Technical Report CS 99-13, http://www.cs.brown.edu/people/pw.

[49] D. Yafaev, *New channels in three-body long-range scattering*, Equations aux derivées partielles, Publ. Ecole Polytechnique, Palaiseau XIV (1994), 1–11.

[50] K. Yosida, Functional Analysis, Springer-Verlag, 1968.